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Null hypersurfaces in generalized Robertson–Walker spacetimes

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ABSTRACT

We study the geometry of null hypersurfaces M in generalized Robertson–Walker spacetimes. First we characterize such null hypersurfaces as graphs of generalized eikonal functions over the fiber and use this characterization to show that such hypersurfaces are parallel if and only if their fibers are also parallel. We further use this technique to construct several examples of null hypersurfaces in both de Sitter and anti de Sitter spaces. Then we characterize all the totally umbilical null hypersurfaces M in a Lorentzian space form (viewed as a quadric in a semi-Euclidean ambient space) as intersections of the space form with a hyperplane. Finally we study the totally umbilical spacelike hypersurfaces of null hypersurfaces in space forms and characterize them as planar sections of M.

1. Introduction

Semi-Riemannian geometry is nowadays a well-established area of research, partly motivated by its applications to General Relativity. Even though semi-Riemannian submanifolds (i.e., those whose induced metric is non-degenerate) have been extensively studied, null submanifolds (with degenerate metric) are less understood, in spite of the fact that numerous features of relevant physical meaning in Relativity find their mathematical grounds in such geometrical objects. That is the case of light trajectories or the smooth parts of event and Cauchy horizons just to name a few.

In spite of their relevance, a systematic study of null submanifolds from a mathematical point of view only flourished from the decade of 1980. Since then, several concepts and results from the semi-Riemannian scenario have been extended to this context, sometimes following different approaches in order to give adequate definitions of the geometrical objects required to study these submanifolds. For example, the Refs. [1–3] provide a broad vision on the subject.

It is also worth noting that the submanifold geometry of null manifolds is yet to be explored in full detail. An example of particular physical interest related to the occurrence of gravitational collapse consists in the study of a spacelike surface S immersed in a null hypersurface M of a four dimensional spacetime $\bar{M}$; see [4–6]. In this setting, S represents the surface of a collapsing massive object (a star for instance) while M is the event horizon associated to the corresponding black hole. Thus, from a geometrical perspective, one of the main problems that arise in this scenario consists in relating the geometrical properties of S and M; in particular, the problem of characterizing the spacelike surfaces subject to suitable geometrical

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restrictions that can be immersed in a null hypersurface of spacetime. A remarkable result in this direction was obtained by Asperti and Dajczer in [7]: For $n \geq 3$, a simply connected n-dimensional Riemannian manifold M may be isometrically immersed in the $(n+1)$-dimensional lightcone if and only if M is conformally flat. As it turns out, even when dealing to the simplest of the null hypersurfaces – namely the light cone in Lorentz–Minkowski space – reveals a rich geometry, as can be seen in the results presented in [8–16]. In particular, in [14], the authors showed that a spacelike hypersurface S of the lightcone in the Minkowski space $\mathbb{R}^{n+2}_1$ is U-totally umbilical with respect to any normal vector field U if and only if S is the intersection of the lightcone with a $(n+1)$-dimensional hyperplane not passing through the origin.

In this paper we will focus in the study of the geometry of null hypersurfaces of a larger class of spacetimes that include all the Lorentzian space forms, namely, the class of generalized Robertson–Walker (GRW) spacetimes. Recall that a GRW spacetime is a Lorentzian warped product $\bar{M} = -I \times_\varrho F$, I being a real interval, F a Riemannian manifold and ϱ a differentiable, real, positive function defined on I. If F has constant sectional curvature, then $\bar{M}$ is called a *Robertson–Walker spacetime*. These spacetimes play a key role in Cosmology since they represent the evolution over time of a homogeneous and isotropic universe [4–6]. The class of Robertson–Walker spacetimes includes the de Sitter space $\mathbb{S}^{n+2}_1$ (when F is a sphere) and the anti de Sitter space $\mathbb{H}^{n+2}_1$ (when F is a hyperbolic space), which together with Lorentz–Minkowski space $\mathbb{R}^{n+2}_1$ encompass the class of Lorentzian space forms.

The present work is organized as follows: In Section 2 we establish the notation, definitions and basic structure equations involving null hypersurfaces and their spacelike hypersurfaces. Then in Section 3 we study the geometry of null parallel hypersurfaces in GRW spacetimes. By proving that a submanifold in a GRW spacetime given as the graph of a function $f : F \to \mathbb{R}$ over the fiber is null if and only if it is the graph of a *generalized eikonal* function defined on the fiber we show that such null hypersurface is parallel if and only if its fiber is also parallel; see Theorem 3.8.

In Section 4 we specialize our study to Robertson–Walker spacetimes and use the techniques developed in the previous section to construct concrete examples. Moreover, in Proposition 4.9 we characterize all the totally umbilical null hypersurfaces M in a Lorentzian space form $\bar{M}$ as the intersections of $\bar{M}$ with a hyperplane.

Finally, in Section 5 (Theorem 5.3) we characterize the totally umbilical spacelike hypersurfaces of a totally umbilical null hypersurface M of $\mathbb{S}^{n+2}_1$ and $\mathbb{H}^{n+2}_1$ as the intersections of M with the totally geodesic hypersurfaces of the corresponding Lorentzian space form, thus extending the results presented in [14].

2. Preliminaries

We will follow closely the notation in [17,1,2]. Let $\bar{M}^{n+2}$ be a $(n+2)$-dimensional, semi-Riemannian manifold with metric $\langle\ ,\ \rangle$ and semi-Riemannian connection $\bar{\nabla}$. A submanifold M of $\bar{M}$ is *null* if the restriction of the metric to M is degenerate at each point $p \in M$, which in turn means that for every such p there is a non-zero vector $\xi_p \in T_pM$ such that $\langle \xi_p, X_p\rangle = 0$ for each $X_p \in T_pM$. As usual, if $\dim M = n+1$, we say that M is a *hypersurface* of $\bar{M}$.

Given a null hypersurface $M \subset \bar{M}$, we will consider a *screen distribution* $S(TM)$, that is, a n-dimensional distribution in TM such that the restriction of the metric of M to $S(TM)$ is positive definite. From [1], we know that in a coordinate neighborhood $U \subset M$ there is a vector field N such that

$$\langle \xi, N\rangle = 1, \qquad \langle N, N\rangle = \langle N, X\rangle = 0 \tag{1}$$

for each $X \in \Gamma(S(TM|_U))$, where ξ is a vector field extension of ξ_p to U. We use ξ and N to decompose the tangent bundle $T\bar{M}$ into three vector bundles. First we write $T\bar{M}$ locally as

$$T\bar{M} = TM \oplus \mathrm{span}(N). \tag{2}$$

Additionally, we express TM as

$$TM = S(TM) \oplus_{\mathrm{orth}} \mathrm{span}(\xi), \tag{3}$$

so that

$$T\bar{M} = S(TM) \oplus_{\mathrm{orth}} (\mathrm{span}(\xi) \oplus \mathrm{span}(N)).$$

Let P be the projection of $\Gamma(TM)$ onto $\Gamma(S(TM))$ using the decomposition (3). The local Gauss–Weingarten formulae are

$$\begin{aligned}
&\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) = \nabla_X Y + B(X,Y)N,\\
&\bar{\nabla}_X N = -A_N X + \nabla^t_X N = -A_N X + \tau(X)N;\\
&\nabla_X PY = \nabla^*_X PY + h^*(X,PY) = \nabla^*_X PY + C(X,PY)\xi;\\
&\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi = -A^*_\xi X - \tau(X)\xi,
\end{aligned} \tag{4}$$

where $X, Y \in \Gamma(TM)$. Here ∇, ∇^t, ∇^* and ∇^{*t} denote the induced connections on TM, $span(N)$, $S(TM)$ and $span(\xi)$, respectively; h and h^* are the second fundamental forms of M and $S(TM)$,

$$B(X,Y) = \langle \bar{\nabla}_X Y, \xi\rangle = \langle A^*_\xi X, Y\rangle,$$

$$C(X,PY) = \langle \nabla_X PY, N\rangle = \langle A_N X, PY\rangle,$$

are the *local second fundamental forms* of M and $S(TM)$, while A_N and A^*_ξ are the *shape operators* on TM and $S(TM)$, respectively. Finally, τ is the 1-form on TM given by $\tau(X) = \langle \bar{\nabla}_X N, \xi\rangle$.

Remark 2.1. Screen distributions are not always integrable. We will focus on submanifolds of null hypersurfaces of $\bar{M}$, so in this case the screen distributions will be the tangent bundles of such submanifolds, and hence the distributions will be integrable. By Theorem 2.3 in [1], integrability is equivalent to the fact of being A_N symmetric on $\Gamma(S(TM))$, that is,

$$\langle A_N X, Y\rangle = \langle X, A_N Y\rangle$$

for each $X, Y \in \Gamma(S(TM))$.

In this paper we will study totally umbilical and parallel submanifolds. Whenever these objects are semi-Riemannian (i.e., the induced metric is non-degenerate), the notions are defined in the usual way. As for null hypersurfaces, we include the definitions for completeness.

Definition 2.2. A null hypersurface M is *totally umbilical* in $\bar{M}$ if there exists a function μ such that $B(X, Y) = \mu\langle X, Y\rangle$ for every $X, Y \in \Gamma(TM)$.

A screen distribution $S(TM)$ is *totally umbilical* in M if there exists a function λ such that $C(X, PY) = \lambda\langle X, PY\rangle$ for every $X, Y \in \Gamma(TM)$. If the function μ (or λ) vanishes identically, we call the corresponding object *totally geodesic*.

Umbilicity may be expressed in terms of the shape operators. It is easily seen that $A^*_\xi \xi = 0$, so that M is totally umbilical if and only if there exists a function μ such that $A^*_\xi X = \mu X$ for every $X \in \Gamma(S(TM))$. On the other hand, we have the following result:

Proposition 2.3 (*Dong–Liu, [18]*). *$S(TM)$ is totally umbilical in M if and only if there is a function λ such that $A_N X = \lambda X$ for every $X \in \Gamma(S(TM))$ and $A_N \xi = 0$.*

Note that the umbilicity of M refers to the operator A^*_ξ; it is known that this definition does not depend on the screen distribution. (See for example [1], p. 107.)

Definition 2.4. If $X, Y, Z \in \Gamma(TM)$, we define

$$\begin{aligned}
&(\nabla_X h)(Y, Z) = \nabla^t_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),\\
&(\nabla_X h^*)(Y, PZ) = \nabla^{*t}_X(h^*(Y, PZ)) - h^*(\nabla_X Y, PZ) - h^*(Y, \nabla^*_X PZ),\\
&(\nabla_X B)(Y, Z) = XB(Y, Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z),\\
&(\nabla_X C)(Y, PZ) = XC(Y, PZ) - C(\nabla_X Y, PZ) - B(Y, \nabla^*_X PZ).
\end{aligned} \tag{5}$$

M is *parallel* if its second fundamental form h is parallel; that is, $\nabla_X h = 0$ for every $X \in \Gamma(TM)$; or equivalently, if $(\nabla_X B)(Y, Z) = -\tau(X)B(Y, Z)$ for every $X, Y, Z \in \Gamma(TM)$.

Similarly, $S(TM)$ is *parallel* if h^* is parallel; that is, $\nabla_X h^* = 0$ for every $X \in \Gamma(TM)$; or equivalently, if $(\nabla_X C)(Y, PZ) = \tau(X)C(Y, PZ)$ for every $X, Y, Z \in \Gamma(TM)$.

We will consider mainly the case when the semi-Riemannian manifold $\bar{M}$ is a *generalized Robertson–Walker spacetime* (or GRW spacetime for short); that is, a Lorentzian warped product of the form $-I \times_\varrho F$, where F is a $(n+1)$-dimensional Riemannian manifold and ϱ is a differentiable, positive function defined in a real interval $I \subset \mathbb{R}$. We recall also that a *Robertson–Walker spacetime* is a GRW spacetime where the fiber F has constant sectional curvature.

In particular, we study submanifolds of the de Sitter and the anti de Sitter spaces, which for completeness we define here. Let $\mathbb{R}^{n+3}_1$ be the $(n+3)$-dimensional vector space with metric

$$\langle x, y\rangle = -x_0 y_0 + \sum_{i=1}^{n+2} x_i y_i,$$

where $x = (x_0, \ldots, x_{n+2})$ and $y = (y_0, \ldots, y_{n+2})$ belong to $\mathbb{R}^{n+3}_1$. Similarly, the metric in semi-Euclidean space $\mathbb{R}^{n+3}_2$ is given by

$$\langle x, y\rangle = -x_0 y_0 - x_1 y_1 + \sum_{i=2}^{n+2} x_i y_i.$$

The $(n+2)$-*dimensional de Sitter space* is the unit sphere in $\mathbb{R}^{n+3}_1$ defined by

$$\mathbb{S}^{n+2}_1 = \{p \in \mathbb{R}^{n+3}_1 \mid \langle p, p\rangle = 1\}.$$

The de Sitter space may be described as a warped product $-\mathbb{R} \times_{\cosh} \mathbb{S}^{n+1}$. In fact, the transformation

$$-\mathbb{R} \times_{\cosh} \mathbb{S}^{n+1} \to \mathbb{S}^{n+2}_1, \qquad (t, p) \mapsto (\sinh t, \cosh t\, p) \tag{6}$$

is an isometry. On the other hand, the $(n+2)$-*dimensional anti de Sitter space* is defined as the hyperquadric

$$\mathbb{H}^{n+2}_1 = \{p \in \mathbb{R}^{n+3}_2 \mid \langle p, p\rangle = -1\}$$

immersed in $\mathbb{R}^{n+3}_2$. An unbounded open region of this hyperquadric can be described as the warped product $-\mathbb{R}\times_{\cos}\mathbb{H}^{n+1}$ via the isometry

$$-\mathbb{R}\times_{\cos}\mathbb{H}^{n+1}\to\mathbb{H}^{n+2}_1,\qquad (t,p)\mapsto(\sin t,\cos t\,p). \tag{7}$$

Let us recall that $\mathbb{S}^{n+2}_1$ and the universal cover of $\mathbb{H}^{n+2}_1$ are the Lorentzian space forms of constant sectional curvature $K=1$ and $K=-1$, respectively.

3. Null hypersurfaces in GRW spacetimes

In this section we will characterize null hypersurfaces as graphs of some special kind of functions, namely, the distance to a fixed submanifold S.

Proposition 3.1. *Let F be a Riemannian manifold and $f:F\to\mathbb{R}$ be a differentiable function. Then the graph of f given as*

$$\{(f(p),p)\,|\,p\in F\}$$

is a null hypersurface in $-I\times_\varrho F$ if and only if

$$|\mathrm{grad}\,f|=\varrho\circ f. \tag{8}$$

Proof. Let e_i, $i=1,\dots,n+1$, be an orthonormal frame field in F. Then a frame field tangent for the graph of f is $E_i=(e_i(f),e_i)$, $i=1,\dots,n+1$. A vector field everywhere normal to the graph is $\xi=((\varrho\circ f)^2,\mathrm{grad}\,f)$. The graph of f is a null hypersurface if and only if ξ is also tangent to the graph, meaning that $\xi=\sum a_iE_i$; in other words,

$$((\varrho\circ f)^2,\mathrm{grad}\,f)=\sum a_i(e_i(f),e_i),$$

implying that $a_i=\langle\mathrm{grad}\,f,e_i\rangle=e_i(f)$; hence,

$$|\,\mathrm{grad}\,f|^2=\sum(e_i(f))^2=(\varrho\circ f)^2.\quad\square$$

Remark 3.2. Functions satisfying $|\,\mathrm{grad}\,f|=C$, where C is constant, are called *eikonal*, while those for which $|\,\mathrm{grad}\,f|=\varrho\circ f$ are called *generalized eikonal functions*. The latter may be characterized as deformations of "signed distance functions" by using some results in [19] (see also [20,21]). In order to precise this statement and for the sake of completeness, let us make a brief review of the techniques applied in the cited references.

Consider a $(n+1)$-dimensional Riemannian manifold F, a hypersurface $S\subset F$, a point $p\in S$ and a normal neighborhood U of p in F. Let $V=U\cap S$ and $Z:V\to TV^{\perp}$ a unit local normal vector field. If $(x_1,\dots,x_n)$ are local coordinates for V, then $(t,x_1,\dots,x_n)\mapsto\exp(tZ(x_1,\dots,x_n))$ defines the usual Fermi coordinates for U. In [19] it is proved that the function $d:U\to\mathbb{R}$ given by $d(\exp(tZ(x_1,\dots,x_n)))=t$ is eikonal; in fact, $|\mathrm{grad}\,d|=1$. We will say that d is the signed distance function associated to the hypersurface S.

Now we are going to construct null hypersurfaces in Lorentzian warped products.

Proposition 3.3. *Let F be a Riemannian manifold, $S\subset F$ a hypersurface, $p\in S$ and $\varrho:I\to\mathbb{R}^+$ a smooth positive function. There is a neighborhood U of p in F and a function $f:U\to\mathbb{R}$ satisfying (8); in consequence, the graph of f is a null hypersurface in $-I\times_\varrho F$.*

Proof. Define a function g by

$$g(s)=\int_{s_0}^{s}\frac{d\sigma}{\varrho(\sigma)},\quad s\in I. \tag{9}$$

Since $g'>0$, g is invertible and $(g^{-1})'(u)=\varrho(g^{-1}(u))$. Note also that its image $J=g(I)$ is an open interval containing 0.

Let $d:U\to\mathbb{R}$ be the function constructed before the statement of this Proposition. By restricting U we may consider that $f=g^{-1}\circ d$ is well defined; hence,

$$|\,\mathrm{grad}\,f|=|((g^{-1})'\circ d)\,\mathrm{grad}\,d|=|\varrho\circ g^{-1}\circ d|=\varrho\circ f.$$

The last statement is a consequence of Proposition 3.1. $\square$

We will also use a converse of the above Proposition.

Proposition 3.4. *Let F be a Riemannian manifold and $\varrho:I\to\mathbb{R}^+$ a smooth positive function. Let $f:F\to\mathbb{R}$ be a function such that the graph of f is a null hypersurface in $-I\times_\varrho F$. For each point $p\in F$ there is a neighborhood U of p in F and a hypersurface $S\subset F$ passing through p such that $f|_U=g^{-1}\circ d$, where g is given by (9) and d is defined by using the Fermi coordinates relative to S.*

Proof. By Proposition 3.1 the function f satisfies $|\operatorname{grad} f| = \varrho \circ f$. Consider the function $d = g \circ f$; we have

$$|\operatorname{grad} d| = |(g' \circ f) \cdot \operatorname{grad} f| = \frac{1}{\varrho \circ f} |\operatorname{grad} f| = 1;$$

hence d is an eikonal function; by applying Theorem 5.3 in [19] we have that d is given locally by the Fermi coordinates relative to some hypersurface S, as claimed. □

The rest of this section is devoted to the parallel null hypersurfaces of a GRW spacetime $\bar{M} = -I \times_\varrho F^{n+1}$. First and for the sake of completeness we give a proof of the following known result.

Lemma 3.5. *Let $\bar{M} = -I \times_\varrho F^{n+1}$ be a GRW spacetime, H_2 the mean curvature vector and h_2 the second fundamental form of a slice $\{t\} \times_\varrho F$ in $\bar{M}$. Then H_2 and h_2 are parallel.*

Proof. Following [17], we know that the slice is totally umbilical:

$$h_2(Y, Z) = -\langle Y, Z\rangle H_2,$$

where the mean curvature vector of a slice is

$$H_2 = -\frac{\varrho'}{\varrho} \partial_t;$$

but also by [17] we have, for every X tangent to the slice,

$$\bar{\nabla}_X H_2 = -\frac{\varrho'}{\varrho} \bar{\nabla}_X \partial_t = -\left(\frac{\varrho'}{\varrho}\right)^2 X,$$

so that $\nabla^\perp_X H_2 = 0$ and hence H_2 is parallel. On the other hand, if X, Y, Z are tangent to the slice, then

$$\begin{aligned}(\nabla_X h_2)(Y, Z) &= \nabla^\perp_X (h_2(Y, Z)) - h_2(\nabla_X Y, Z) - h_2(Y, \nabla_X Z)\\ &= -\nabla^\perp_X (\langle Y, Z\rangle H_2) + \langle \nabla_X Y, Z\rangle H_2 + \langle Y, \nabla_X Z\rangle H_2\\ &= -\langle Y, Z\rangle \nabla^\perp_X H_2 = 0,\end{aligned}$$

and so H_2 and h_2 are parallel. □

Recall from Section 2 that for every null hypersurface M we define the 1-form τ on TM by $\tau(X) = \langle \bar{\nabla}_X N, \xi\rangle$.

Lemma 3.6. *Let $\bar{M} = -I \times_\varrho F^{n+1}$ be a GRW spacetime, M be a null hypersurface in $\bar{M}$ given as the graph of a function f, and $S(TM)$ the screen distribution given by the level hypersurfaces of f as in Section 3. Then $\tau(X) = 0$ for any $X \in \Gamma(S(TM))$.*

Proof. Following the notation of Section 3, we have for any $X \in \Gamma(S(TM))$,

$$\bar{\nabla}_X N = \frac{1}{\sqrt{2}} \bar{\nabla}_X (E_{n+1} - \partial_t) = \frac{1}{\sqrt{2}} \left(\bar{\nabla}_X E_{n+1} - \frac{\varrho'}{\varrho} \partial_t\right),$$

where again we use the formulae in [17]. We have

$$\langle \bar{\nabla}_X N, \xi\rangle = \frac{1}{\sqrt{2}} \langle \bar{\nabla}_X E_{n+1}, \xi\rangle = \frac{1}{2} \langle \bar{\nabla}_X E_{n+1}, E_{n+1} + \partial_t\rangle.$$

As E_{n+1} is a unit vector, $\langle \bar{\nabla}_X E_{n+1}, E_{n+1}\rangle = \frac{1}{2} X\langle E_{n+1}, E_{n+1}\rangle = 0$; on the other hand,

$$\langle \bar{\nabla}_X E_{n+1}, \partial_t\rangle = -\langle E_{n+1}, \bar{\nabla}_X \partial_t\rangle = -\left\langle E_{n+1}, \frac{\varrho'}{\varrho} X\right\rangle = 0;$$

in short, $\tau(X) = \langle \bar{\nabla}_X N, \xi\rangle = 0$ for $X \in \Gamma(S(TM))$. □

Corollary 3.7. *For $X \in \Gamma(S(TM))$ we have*

$$\nabla^t_X N = 0, \qquad \nabla^{*t}_X \xi = 0 \tag{10}$$

or, equivalently,

$$\bar{\nabla}_X N = -A_N X, \qquad \bar{\nabla}_X \xi = -A^*_\xi X. \tag{11}$$

We end this section considering the case of a null hypersurface in a GRW spacetime and relating it with its fiber.

Theorem 3.8. *Let $\bar{M} = -I \times_\varrho F^{n+1}$ be a GRW spacetime, M be a null hypersurface in $\bar{M}$ given as the graph of a function f. If the second fundamental form h of M is parallel, then f is the signed distance function associated to a parallel hypersurface S in F^{n+1}.*

Proof. Since h is parallel,

$$(\nabla_X B)(Y, Z) = -\tau(X)B(Y, Z)$$

for every $X, Y, Z \in \Gamma(TM)$. By Lemma 3.6, $\nabla_X B = 0$ for $X \in \Gamma(S(TM))$.

Let $S(TM)$ be the screen distribution given by the level hypersurfaces of f as in Section 5. Using again Lemma 3.6, $\nabla_X C = 0$ for $X \in \Gamma(S(TM))$.

As in the previous section, let h_1 and B_1 be the associated second fundamental forms of $S_t = \{t\} \times_\varrho f^{-1}(t)$ relative to the slice $\{t\} \times_\varrho F$. We have

$$\begin{aligned} B_1(Y, Z) &= \langle \bar{\nabla}_Y Z, E_{n+1}\rangle \\ &= \frac{1}{\sqrt{2}}\left(\langle \bar{\nabla}_Y Z, \xi\rangle + \langle \bar{\nabla}_Y Z, N\rangle\right) = \frac{1}{\sqrt{2}}\left(B(Y, Z) + C(Y, Z)\right) \end{aligned}$$

for $X, Y, Z \in \Gamma(S(TM))$. Hence $\nabla_X B_1 = (\nabla_X B + \nabla_X C)/\sqrt{2} = 0$.

Since $h_1(Y, Z) = B_1(Y, Z)E_{n+1}$,

$$(\nabla_X h_1)(Y, Z) = (\nabla_X B_1)(Y, Z) + B_1(Y, Z)\nabla^\perp_X E_{n+1} = B_1(Y, Z)\nabla^\perp_X E_{n+1};$$

where $\nabla^\perp$ is the normal connection of S_t in $\{t\} \times_\varrho F$. But $\langle \nabla^\perp_X E_{n+1}, E_{n+1}\rangle = 0$; hence $\nabla_X h_1 = 0$ and S_t is parallel in F. □

4. Examples of null hypersurfaces in $\mathbb{S}^{n+2}_1$ and $\mathbb{H}^{n+2}_1$

Let us apply the methods given in Section 3 to get some examples of null hypersurfaces in both de Sitter and anti de Sitter spaces. We begin with the case of $\mathbb{S}^{n+2}_1$. We recall that this space is isometric to the warped product $-\mathbb{R} \times_{\cosh} \mathbb{S}^{n+1}$ under the mapping $(t, p) \mapsto (\sinh t, \cosh t \cdot p)$ given in (6).

Example 4.1. Consider the function on $\mathbb{S}^{n+1}$ measuring the signed distance to the equator $\mathbb{S}^n \times \{0\}$; explicitly,

$$d(p) = \frac{\pi}{2} - \cos^{-1}\langle e_{n+2}, p\rangle.$$

Since the warping function is cosh, the function g in (9) is

$$g(s) = \arctan(\sinh(s))$$

and so $g^{-1} = \sinh^{-1} \circ \tan(s)$. Then, $f(p) = g^{-1} \circ d(p)$ is given by

$$\sinh^{-1} \circ \tan\left(\frac{\pi}{2} - \cos^{-1}\langle e_{n+2}, p\rangle\right) = \sinh^{-1} \frac{\langle e_{n+2}, p\rangle}{\sqrt{1 - \langle e_{n+2}, p\rangle^2}}.$$

Using the isometry (6), the graph of f is mapped onto the set of points in $\mathbb{S}^{n+1}_1 \subset \mathbb{R}^{n+3}_1$ of the form

$$\left(\frac{\langle e_{n+2}, p\rangle}{\sqrt{1 - \langle e_{n+2}, p\rangle^2}}, \cosh \circ \sinh^{-1} \frac{\langle e_{n+2}, p\rangle}{\sqrt{1 - \langle e_{n+2}, p\rangle^2}}\, p\right), \quad p \in \mathbb{S}^{n+1},$$

which simplifies to

$$\left(\frac{\langle e_{n+2}, p\rangle}{\sqrt{1 - \langle e_{n+2}, p\rangle^2}}, \frac{1}{\sqrt{1 - \langle e_{n+2}, p\rangle^2}}\, p\right).$$

Since the ith coordinate of p is precisely $\langle e_i, p\rangle$, the above expression shows that this null hypersurface may be characterized also as the set of points in $\mathbb{S}^{n+2}_1$ such that

$$x_0 - x_{n+2} = 0,$$

thus M lies in a hyperplane. Note that this implies that

$$\sum_{i=1}^{n+1} x_i^2 = 1,$$

which let us parametrize the hypersurface by

$$\Phi(s, u_1, \ldots, u_n) = (s, \varphi(u_1, \ldots, u_n), s),$$

where φ is an orthogonal parametrization of the sphere. Also, $\partial\Phi/\partial s$ is a vector field everywhere tangent and normal to our hypersurface.

Example 4.2. The above example is a particular case of the following: Consider the hypersurface $S \subset \mathbb{S}^{n+1}$ given as a "parallel" in $\mathbb{S}^{n+1}$, that is, the set of points making an angle α with the fixed vector e_{n+2}. The signed distance function $d : \mathbb{S}^{n+1} \to \mathbb{R}$ is

$$d(p) = \cos^{-1}\alpha - \cos^{-1}\langle e_{n+2}, p\rangle.$$

Now

$$f(p) = g^{-1} \circ d(p) = \sinh^{-1} \circ \tan\left(\cos^{-1}\alpha - \cos^{-1}\langle e_{n+2}, p\rangle\right).$$

Using the isometry (6) and some calculations, we obtain that the graph of f is mapped onto the set of points in $\mathbb{S}^{n+2}_1 \subset \mathbb{R}^{n+3}_1$ of the form

$$\frac{\left(\langle e_{n+2}, p\rangle\sqrt{1-\alpha^2} - \alpha\sqrt{1-\langle e_{n+2}, p\rangle^2}, p\right)}{\alpha\langle e_{n+2}, p\rangle + \sqrt{1-\alpha^2}\sqrt{1-\langle e_{n+2}, p\rangle^2}}.$$

As in Example 4.1, we may see that this set is the intersection of a hyperplane with $\mathbb{S}^{n+2}_1$. Since $p = \sum\langle e_i, p\rangle e_i$, if x_0 and x_{n+2} denote the first and last coordinates of the above expression, we check that

$$x_{n+2} - \sqrt{1-\alpha^2}\, x_0 = \alpha,$$

which is the equation of a hyperplane. This null hypersurface can be parametrized as

$$\Phi(s, u_1, \ldots, u_n) = (s, R(s)\varphi(u_1, \ldots, u_n), \sqrt{1-\alpha^2}\, s + \alpha), \tag{12}$$

where $R(s) = \alpha s - \sqrt{1-\alpha^2}$ and φ is an orthogonal parametrization of $\mathbb{S}^n$. Here

$$\frac{\partial\Phi}{\partial s} = (1, \alpha\varphi(u_1, \ldots, u_n), \sqrt{1-\alpha^2})$$

is a null vector field tangent to M. Notice that Example 4.1 corresponds to the case $\alpha = 0$, as expected.

Example 4.3. Although the present example does not fall precisely into the category of a function measuring the distance to a hypersurface, it can be considered as a limit case of the previous ones. We take the function in $\mathbb{S}^{n+1}$ measuring the distance to a fixed point, say e_i, where $e_1, \ldots, e_{n+2}$ is the canonical orthonormal basis of $\mathbb{R}^{n+2}$. Explicitly, $d : \mathbb{S}^{n+1} \to \mathbb{R}$ is given by $d(p) = \cos^{-1}\langle e_i, p\rangle$. The graph of $f = g^{-1} \circ d$, i.e., the set

$$(\sinh^{-1} \circ \tan \circ \cos^{-1}\langle e_i, p\rangle, p), \quad p \in \mathbb{S}^{n+1},$$

is a null hypersurface in the warped product $-\mathbb{R} \times_{\cosh} \mathbb{S}^{n+1}$. In order to get an easier description, we use again the isometry (6) to obtain that the above graph is mapped onto the set

$$(\tan \circ \cos^{-1}\langle e_i, p\rangle, \cosh \circ \sinh^{-1} \circ \tan \circ \cos^{-1}\langle e_i, p\rangle \cdot p).$$

We use the identity $\cosh \circ \sinh^{-1} \circ \tan x = 1/\cos x$ to simplify this as

$$\left(\tan \circ \cos^{-1}\langle e_i, p\rangle, \frac{1}{\langle e_i, p\rangle} p\right).$$

Since we considered the canonical basis, the ith coordinate of the above point is equal to 1, which means that the null hypersurface is precisely the set $M = \mathbb{S}^{n+2}_1 \cap \{x_i = 1\}$.

In order to complete the description of M, let us give its tangent vectors. For simplicity, set $i = n + 2$ and consider the set $\mathbb{S}^{n+2}_1 \cap \{x_{n+2} = 1\}$. Since the points of the de Sitter space satisfy

$$-x_0^2 + \sum_{i=1}^{n+2} x_i^2 = 1$$

and $x_{n+2} = 1$, we have

$$-x_0^2 + \sum_{i=1}^{n+1} x_i^2 = 0,$$

which is a $(n + 1)$-dimensional lightcone in $\mathbb{R}^{n+2}_1$. We parametrize this cone by

$$(s, u_1, \ldots, u_n) \mapsto s(1, \varphi(u_1, \ldots, u_n)),$$

where φ is an orthogonal parametrization of a unit n-dimensional sphere. Finally, our null hypersurface is parametrized by

$$\Phi(s, u_1, \dots, u_n) = (s, s\varphi(u_1, \dots, u_n), 1).$$

Now it is easy to see that the vector fields

$$\frac{\partial \Phi}{\partial s}, \frac{\partial \Phi}{\partial u_1}, \dots, \frac{\partial \Phi}{\partial u_n},$$

span TM at each point, that they are orthogonal and that the first one is null.

Let us focus now in the anti de Sitter space $\mathbb{H}_1^{n+2}$. We will work in the region isometric to the warped product $-\mathbb{R} \times_{\cos} \mathbb{H}^{n+1}$ to construct null hypersurfaces similarly to those presented in Example 4.2. Since we will be dealing with hypersurfaces immersed in the hyperbolic space $\mathbb{H}^{n+1}$, we center our attention in three distinct families of hypersurfaces: geodesic spheres, horospheres and equidistant surfaces. The following lemma will be useful in handling computations.

Lemma 4.4. *Let $d : \mathbb{H}^{n+1} \to \mathbb{R}$ be the signed distance function to a hypersurface $S \subset \mathbb{H}^{n+1}$. If d is of the form $d(p) = \cosh^{-1}\alpha - \cosh^{-1}\beta$, then the graph of $f = g^{-1} \circ d$ in $\mathbb{H}_1^{n+2} \subset \mathbb{R}_2^{n+2}$ is the set*

$$\frac{\left(\beta\sqrt{1-\alpha^2} - \alpha\sqrt{\beta^2 - 1}, p\right)}{\alpha\beta + \sqrt{\alpha^2 - 1}\sqrt{\beta^2 - 1}}, \quad p \in \mathbb{H}^{n+1}.$$

Proof. Since the warping function in this case is $\rho(\sigma) = \cos\sigma$, we have that $g(s) = \int_0^s \frac{d\sigma}{\rho(\sigma)} = \sinh^{-1}(\tan(s))$ and thus

$$f(p) = \arctan\sinh(\cosh^{-1}\alpha - \cosh^{-1}\beta) = \arctan(\beta\sqrt{\alpha^2 - 1} - \alpha\sqrt{\beta^2 - 1}).$$

Hence, under the isometry (7) the graph of f maps to the set $(\sin f(p), p\cos(p))$ which is precisely the desired set. $\square$

Example 4.5. Let us consider a geodesic sphere S in the space $\mathbb{H}^{n+1}$ and let $\{e_1, e_2, \dots, e_{n+2}\}$ be an orthonormal basis of $\mathbb{R}_1^{n+2}$ with timelike e_1. After a suitable hyperbolic isometry we can realize S as a hypersurface making a constant hyperbolic angle $s_0 = \cosh^{-1}\alpha$ relative to the fixed timelike vector e_1, that is, a geodesic sphere. Thus S is a hypersurface that represents a "parallel" in the hyperboloid $\mathbb{H}^{n+1} \subset \mathbb{R}_1^{n+2}$. Notice that in this case S is the intersection of $\mathbb{H}^{n+1}$ with a spacelike hyperplane. Thus the signed distance d: $\mathbb{H}^{n+1} \to \mathbb{R}$ is given by

$$d(p) = \cosh^{-1}\alpha - \cosh^{-1}(-\langle e_1, p\rangle)$$

and hence by virtue of Lemma 4.4 the graph of f is equivalent to the locus in $\mathbb{H}_1^{n+2} \subset \mathbb{R}_2^{n+3}$ of points of the form

$$\frac{\left(-\langle e_1, p\rangle\sqrt{\alpha^2 - 1} - \alpha\sqrt{\langle e_1, p\rangle^2 - 1}, p\right)}{\alpha\sqrt{\langle e_1, p\rangle} + \sqrt{\alpha^2 - 1}\sqrt{\langle e_1, p\rangle}}.$$

It follows, just as in Example 4.2, that this set is contained in the intersection of $\mathbb{H}_1^{n+2}$ with the hyperplane

$$\sqrt{\alpha^2 - 1}\, x_0 + x_1 = -\alpha.$$

In fact, a parametrization of M can be given by

$$\Phi(s, u_1, \dots, u_n) = (s, -\alpha - \sqrt{\alpha^2 - 1}\, s, R(s)\varphi(u_1, \dots, u_n)), \tag{13}$$

where $R(s) = \sqrt{\alpha^2 - 1} + \alpha s$ and φ is an orthogonal parametrization of $\mathbb{S}^n$. Notice that the vector field

$$\frac{\partial \Phi}{\partial s} = (1, -\sqrt{\alpha^2 - 1}, \varphi(u_1, \dots, u_n), -\alpha)$$

is null and tangent to M.

Example 4.6. Horospheres can be dealt in a similar fashion than geodesic spheres. In fact, we arrive at an equivalent form of Eq. (13). To this end, let us consider the Poincaré model of upper-half space and let us compute the signed distance d between a given point $\hat{p}$ and a horosphere $\hat{S}$. Without loss of generality, we can suppose that $\hat{p} = (0, \dots, 0, y)$ and that $\hat{S}$ is the horizontal hyperplane $\hat{S} = (0, \dots, 0, c)$ with constant c. Thus $d = \ln c - \ln y$. Now, by means of a standard isometry (cfr. [22], chapter 3) $\hat{p}$ is mapped to the point $p = ((y^2 + 1)/2y, 0, \dots, 0, (y^2 - 1)/2y) \in \mathbb{H}^{n+1}$ whereas $\hat{S}$ is mapped to the hyperplane $x_1 - x_{n+2} = 1/c$. A straightforward computation shows that $d = \ln c - \cosh^{-1} x_1$. By choosing α such that $\cosh^{-1}\alpha = \ln c$ we arrive to the same form for d as in the previous example.

Example 4.7. We now consider a different scenario in which S is the result of intersecting $\mathbb{H}^{n+1}$ with a timelike hyperplane. Thus we can think of S as a hypersurface parallel to a equidistant surface (that is, a "meridian") of $\mathbb{H}^{n+1}$ given by $s_0 = \cosh^{-1}\alpha$. In order to calculate the distance between S and $p \in \mathbb{H}^{n+1}$ we first find the distance from both to the meridian $M_0 = \{x_{n+2} = 0\}$. For any p, consider the point p' which corresponds to the hyperbolic reflection of p with respect to M_0. Then the geodesic $\mathbb{H}^{n+1} \cap span\{p, p'\}$ that joins p to p' is orthogonal to M_0. As a consequence, half of its arc length gives us the desired distance. Then we can show that $\langle e_{n+2}, p\rangle = p_{n+2} = \sinh s$, where s is the distance from p to M_0. Thus, after using the isometry (7) we can find that M is the set of points of the form

$$\frac{\left(\sqrt{\alpha^2-1}\sqrt{\langle e_{n+2}, p\rangle^2+1}+\alpha, p\langle e_{n+2}, p\rangle\right)}{\alpha\sqrt{\langle e_1, p\rangle^2-1}+\sqrt{\alpha^2-1}\langle e_{n+2}, p\rangle},$$

namely, it is a portion of the intersection of $\mathbb{H}^{n+2}_1$ with the hyperplane

$$\alpha x_0 + x_{n+2} = \sqrt{\alpha^2-1}.$$

In this case, we can parametrize the null hypersurface M as

$$\Phi(s, u_1, \dots, u_n) = (s, R(s)\psi(u_1, \dots, u_n), \sqrt{\alpha^2-1} - \alpha s), \tag{14}$$

where $R(s) = \sqrt{\alpha^2-1}s - \alpha$ and ψ is an orthogonal parametrization of $\mathbb{H}^n$. Here we also have that the vector field

$$\frac{\partial\Phi}{\partial s} = (1, \sqrt{\alpha^2-1}\psi(u_1, \dots, u_n), -\alpha).$$

is null.

The above examples are not only illustrative of our technique for constructing null hypersurfaces in GRW spacetimes, but they also provide examples of totally umbilical and totally geodesic null hypersurfaces.

Proposition 4.8. *Let $M \subset \mathbb{S}^{n+2}_1$ be a null hypersurface parametrized by Eq. (12), then M is totally umbilical. Furthermore, M is totally geodesic if and only if $\alpha = 0$. Similarly, if $M \subset \mathbb{H}^{n+2}_1$ is a null hypersurface parametrized by either Eq. (13) or (14), then M is totally umbilical. Furthermore, in the latter case, M is totally geodesic if and only if $\alpha = 1$.*

Proof. Recall that in all these cases we have that $\partial\Phi/\partial s$ is a null vector field tangent to M. Let then $\xi = (1/\sqrt{2})\partial\Phi/\partial s$ and $N = \xi - (2/\sqrt{2})e_0$. Hence ξ and N are null vector fields such that $\langle\xi, N\rangle = 1$ and both of them are orthogonal to the level manifold $S = \Phi(s_0, u_1, \dots, u_n)$ corresponding to $s = s_0$. Thus in this case the screen distribution is given by $S(TM) = span\{\partial\Phi/\partial u_1, \dots, \partial\Phi/\partial u_n\}$. Let $X \in TM$ and denote by D the Levi-Civita connection in the ambient semi-Euclidean space $\mathbb{R}^{n+3}_s$, $s = 1, 2$. Then a straightforward computation shows that

$$D_X\xi = -\mu PX, \quad \forall X \in TM, \tag{15}$$

where

$$\mu = \begin{cases} -\dfrac{\alpha}{\sqrt{2}R}, & \text{for Eqs. (12) and (13),} \\ -\dfrac{\sqrt{\alpha^2-1}}{\sqrt{2}R}, & \text{for Eq. (14).} \end{cases} \tag{16}$$

Thus we have $\langle D_X\xi, \Phi\rangle = 0$ and $\bar{\nabla}_X\xi = D_X\xi$. On the other hand, $B(X, \xi) = 0$ implies that $\bar{\nabla}_X\xi = \nabla_X\xi$, hence

$$\tau(X) = \langle\bar{\nabla}_X N, \xi\rangle = -\langle\bar{\nabla}_X\xi, N\rangle = -\langle\nabla_X\xi, N\rangle = \mu\langle PX, N\rangle = 0. \tag{17}$$

Finally, this last equation coupled with the Gauss–Weingarten formulae (4) yields

$$A^*_\xi(X) = -\nabla_X\xi = \mu PX, \quad \forall X \in TM.$$

Therefore M is totally umbilical as claimed. Notice further that $A_\xi \equiv 0$ precisely when $\alpha = 0$ in the de Sitter case (12), or else $\alpha = 1$ in the anti de Sitter case (14), thus proving the claim. □

The next result shows that these examples are in fact the only totally umbilical null hypersurfaces immersed in either de Sitter or anti de Sitter space. We mention that similar results were obtained with different techniques by Akivis and Goldberg (cfr. Theorem 13 in [23]) and also by Gutiérrez and Olea (cfr. Theorem 4.15 in [24]). Further notice that the totally geodesic cases described in Propositions 4.8 and 4.9 correspond to the ones obtained by Ferrández, Giménez and Lucas in [25].

Proposition 4.9. *Let $\mathbb{K}_1^{n+2}$ denote either $\mathbb{S}_1^{n+2}$ or the region of $\mathbb{H}_1^{n+2}$ isometric to the warped product $-\mathbb{R}\times_{\cos}\mathbb{H}^{n+1}$. Let M be a null, connected and totally umbilical hypersurface of $\mathbb{K}_1^{n+2}$. Then M is planar, i.e. it is contained in the intersection of $\mathbb{K}_1^{n+2}$ with a hyperplane in the ambient semi-Euclidean space $\mathbb{R}_s^{n+3}$, $s = 1, 2$.*

Proof. Using the isometries (6) or (7) we may work in the realization of $\mathbb{K}_1^{n+2}$ as a Robertson–Walker spacetime. Thus let us denote by $\mathbb{K}_0^{n+1}$ the fiber in each case. For simplicity we denote also by M the image of the hypersurface given in the statement of the proposition under this isometry.

Consider the height function $f : \mathbb{K}_1^{n+2} \to \mathbb{R}, f(t,p) = t$. This is a semi-Riemannian totally umbilical submersion, hence by Proposition 4 in [25],

$$\Sigma_t = M \cap f^{-1}(t)$$

is a complete totally umbilical hypersurface in $f^{-1}(t)$. By Proposition 3.4, M is given locally as the graph of the modified signed distance function $g^{-1}\circ d$ to a hypersurface $S\subset\mathbb{K}_0^{n+1}$. Note that each level set of $g^{-1}\circ d$ is contained in precisely one Σ_t. By means of an isometry we may suppose that this sphere is precisely that of Examples 4.2, 4.5, 4.6 or 4.7. As in those Examples, we use again the warping isometry and conclude that M is locally the intersection of $\mathbb{K}_1^{n+2}$ with a hyperplane in $\mathbb{R}_s^{n+3}$. The connectedness of M gives the desired result. □

5. Totally umbilical spacelike submanifolds of null hypersurfaces

We now study the submanifold geometry of a null hypersurface M in $\bar{M} = -I\times_\varrho F$, given as the graph of a function $f : F \to \mathbb{R}$.

For each $t\in\mathbb{R}$, let S_t be the level hypersurface $\{t\}\times f^{-1}(t)\subset M$ and $S(TM)$ be the screen distribution on M given by the tangent bundles of these level hypersurfaces. We consider an orthonormal frame $e_1,\dots,e_n,e_{n+1}$ in $\{t\}\times F$ adapted to S_t, that is, the first n vector fields are tangent to S_t and e_{n+1} is just the unit vector field in the direction of the gradient of f. Define the vector fields $E_i = (0, e_i)\in\Gamma(T\bar{M})$, $i = 1,\dots,n+1$,

$$\xi = \frac{1}{\sqrt{2}}(1, e_{n+1}) \quad\text{and}\quad N = \frac{1}{\sqrt{2}}(-1, e_{n+1}).$$

It is easy to see that ξ is a null vector field tangent to the graph of f and N satisfies (1). Note also that

$$\partial_t = \frac{1}{\sqrt{2}}(\xi - N) \quad\text{and}\quad E_{n+1} = \frac{1}{\sqrt{2}}(\xi + N).$$

In (4) we defined the second fundamental forms h and h^* of M and $S(TM)$. Denote by h_1 and h_2 the second fundamental forms of S_t in $\{t\}\times F$ and of $\{t\}\times F$ in $\bar{M}$, respectively. If $X, Y\in\Gamma(S(TM))$, $h_1(X,Y)$ (resp. $h_2(X,Y)$) is simply the component of $\bar{\nabla}_X Y$ in the direction of E_{n+1} (resp. ∂_t). We have

$$(h + h^*)(X,Y) = (h_1 + h_2)(X,Y)$$

for $X, Y\in\Gamma(S(TM))$. In terms of the shape operators, the equalities

$$A_{E_{n+1}}X = \frac{1}{\sqrt{2}}(A_\xi^* + A_N)X \quad\text{and}\quad A_{\partial_t}X = \frac{1}{\sqrt{2}}(A_\xi^* - A_N)X \tag{18}$$

hold for every $X\in\Gamma(S(TM))$. Recall also that in any warped product, $\{t\}\times F$ is totally umbilical in $\bar{M}$; in fact,

$$A_{\partial_t}X = \frac{\varrho'}{\varrho}X, \tag{19}$$

for each X in $\Gamma(S(TM))$.

Lemma 5.1. $A_N\xi = 0$.

Proof. From [2], p. 48, we know that A_N is $\Gamma(S(TM))$-valued, so in order to show the claim we calculate only the coefficients of $A_N\xi$ relative to the frame E_i, $i = 1,\dots,n$, namely, $\langle A_N\xi, E_i\rangle = -\langle\bar{\nabla}_\xi N, E_i\rangle$. Writing

$$\xi = \frac{1}{\sqrt{2}}(E_{n+1} + \partial_t) \quad\text{and}\quad N = \frac{1}{\sqrt{2}}(E_{n+1} - \partial_t),$$

we express $\bar{\nabla}_\xi N$ in terms of E_{n+1} and ∂_t. By using the standard formulae for a connection in a warped product (see [17]), we have

$$\bar{\nabla}_\xi N = \frac{1}{2}\bar{\nabla}_{E_{n+1}}E_{n+1},$$

but in the slice $\{t\}\times F$, E_{n+1} is a gradient vector field with constant norm; then it is known that its trajectories are geodesics, so the projection of $\bar{\nabla}_{E_{n+1}}E_{n+1}$ in $S(TM)$ vanishes and in consequence $A_N\xi = 0$. □

Proposition 5.2. *Let $\bar{M} = -I \times_{\varrho} F$ be a generalized Robertson–Walker spacetime, M a null hypersurface given as the graph of f as in Section 3. Let $S(TM)$ be the screen distribution defined by the horizontal slices $\{t\} \times f^{-1}(t)$. Then M is totally umbilical in $\bar{M}$ if and only if $S(TM)$ is totally umbilical in M.*

Proof. We compare the shape operators A^*_ξ and A_N. $A^*_\xi \xi = 0$ holds always and by Lemma 5.1, $A_N \xi = 0$, so we just look at their behavior in $S(TM)$; but by Eqs. (18) and (19),

$$\frac{1}{\sqrt{2}}(A^*_\xi - A_N)X = A_{\partial_t}X = \frac{\varrho'}{\varrho}X$$

for $X \in \Gamma(S(TM))$; therefore, A^*_ξ is a multiple of the identity in $S(TM)$ if and only if the same happens for A_N. □

To close this section we study the specific cases of $\mathbb{S}^{n+2}_1$ and $\mathbb{H}^{n+2}_1$, extending Proposition 4.1 in [14] to characterize the totally umbilical spacelike submanifolds of these space forms as planar sections.

Theorem 5.3. *Let $\mathbb{K}^{n+2}_1$ denote either $\mathbb{S}^{n+2}_1$ or the region of $\mathbb{H}^{n+2}_1$ isometric to the warped product $-\mathbb{R} \times_{\cos} \mathbb{H}^{n+1}$. Consider a totally umbilical null hypersurface M immersed in $\mathbb{K}^{n+2}_1$ and let $S \subset M$ be a spacelike hypersurface of M. Then S is totally umbilical in M if and only if S is the intersection of M with a totally geodesic hypersurface of $\mathbb{K}^{n+2}_1$.*

Proof. According to Proposition 4.9 the totally geodesic hypersurfaces of $\mathbb{K}^{n+2}_1$ – whether semi-Riemannian or null – are precisely the intersections of $\mathbb{K}^{n+2}_1$ with hyperplanes through the origin of $\mathbb{R}^{n+3}_s$, $s = 1, 2$. If M is totally geodesic itself, then there is nothing to prove. Otherwise, let us consider S as the intersection of M with a hyperplane, as it was done in the proof of Proposition 4.8. Then notice that in virtue of Eq. (15) we have that

$$\bar{\nabla}_X N = D_X N = D_X \xi = \mu PX, \quad \forall X \in TM.$$

Thus, from Eq. (17) and the Gauss–Weingarten formulae (4) it follows that

$$A_N(X) = -\bar{\nabla}_X \xi = \mu PX, \quad \forall X \in TM,$$

hence S is totally umbilical in M.

In order to show the converse, let us denote by $\nabla^\perp$ the normal connection of S as a codimension three submanifold of $\mathbb{R}^{n+3}_s$ and by $\bar{h}$ its second fundamental form. Now notice that $TS^\perp = span\{\xi, N, \Phi\}$, from which

$$\nabla^\perp_X \xi = \langle \nabla^\perp_X \xi, N\rangle \xi + \langle \nabla^\perp_X \xi, \xi\rangle N + \langle \nabla^\perp_X \xi, \Phi\rangle \Phi.$$

Moreover, since $\nabla^\perp_X \Phi = 0$ then $\langle \xi, \Phi\rangle = 0$ implies $\langle \nabla^\perp_X \xi, \Phi\rangle = 0$. Further, from $\langle \xi, \xi\rangle = 0$ we have $\langle \nabla^\perp_X \xi, \xi\rangle = 0$. Thus

$$\nabla^\perp_X \xi = \langle \nabla^\perp_X \xi, N\rangle \xi.$$

A similar calculation shows that

$$\nabla^\perp_X N = \langle \nabla^\perp_X N, \xi\rangle N.$$

Since $S \subset M$ is totally umbilical, there exists λ such that $A_N X = \lambda X$ for all $X \in \Gamma(TS)$. Furthermore, since $\mathbb{K}^{n+2}_1$ has nonvanishing curvature, Theorem 5.4 in [1] establishes that $\lambda \neq 0$. Thus

$$\begin{aligned}\langle \bar{h}(X, Y), N\rangle &= \langle \bar{\nabla}_X Y, N\rangle = -\langle \bar{\nabla}_X N, Y\rangle \\ &= \langle \lambda X - \langle \nabla^\perp_X N, \xi\rangle N, Y\rangle = \lambda\langle X, Y\rangle\end{aligned}$$

for all $X, Y \in \Gamma(TS)$. Similarly, since M is totally umbilical in $\mathbb{K}^{n+2}_1$, we have that $A^*_\xi X = \mu X$ and thus

$$\langle \bar{h}(X, Y), \xi\rangle = \mu\langle X, Y\rangle.$$

Hence the mean curvature vector $\bar{H}$ is given by $\bar{H} = \lambda N + \mu\xi$ and satisfies $\bar{h}(X, Y) = \langle X, Y\rangle \bar{H}$. If M is totally geodesic, the result follows immediately. Alternatively, $\mu \neq 0$ and $\bar{H}$ is not null. Let us consider then an orthonormal basis of $T\mathbb{K}^{n+2}_1$, $\{\bar{e}_1, \ldots, \bar{e}_n, \bar{e}_{n+1}, \bar{e}_{n+2}\}$ where $\{\bar{e}_1, \ldots, \bar{e}_n\}$ is a basis of TS and $\bar{H} = \bar{e}_{n+1}$, then for all $i = 1, \ldots, n$ we have

$$\langle \bar{\nabla}_X \bar{e}_i, \bar{e}_{n+2}\rangle = \langle \bar{h}(X, \bar{e}_i), \bar{e}_{n+2}\rangle = \langle X, \bar{e}_i\rangle \langle \bar{H}, \bar{e}_{n+2}\rangle = 0.$$

As a consequence, the distribution generated by $\{\bar{e}_1, \ldots, \bar{e}_n, \bar{e}_{n+1}\}$ is parallel along S. Since $\mathbb{K}^{n+2}_1$ has constant curvature, a classical result (refer for instance to Corollary 11 in [26]) implies that S lies in a totally geodesic hypersurface of $\mathbb{K}^{n+2}_1$. □

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