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Constant mean curvature hypersurfaces with constant angle in semi-Riemannian space forms

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ABSTRACT

We study constant angle semi-Riemannian hypersurfaces M immersed in semi-Riemannian space forms, where the constant angle is defined in terms of a closed and conformal vector field Z in the ambient space form. We show that such hypersurfaces belong to the class of hypersurfaces with a canonical principal direction. This property is a type of rigidity. We further specialize to the case of constant mean curvature (CMC) hypersurfaces and characterize them in two relevant cases: when the hypersurface is orthogonal to Z then it is totally umbilical, whereas if Z is tangent to the hypersurface then it has zero Gauss–Kronecker curvature and either its mean curvature vanishes or the ambient is a semi-Euclidean space. We also treat in detail the surface case, giving a full characterization of the constant angle CMC surfaces immersed in all three dimensional space forms. They are isoparametric surfaces with constant principal curvatures when the ambient is flat. If the mean curvature of the surface is not $\pm 2/\sqrt{3}$ they are either totally umbilic or totally geodesic. In particular when the surface has zero mean curvature it is totally geodesic.

1. Introduction

The study of the geometry of constant angle surfaces is as old as the classical theory of surfaces itself. As its name suggests, constant angle surfaces are immersed surfaces $M^2 \subset \mathbb{R}^3$ which make a constant angle with respect to a distinguished vector field Z. These surfaces can be thought as two dimensional analogues of well known curves that make constant angle with a prescribed direction, such as helices (fixed direction), logarithmic spirals (radial direction) or rhumb lines (direction given by meridians on the sphere). In this

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context, one of the most fundamental questions in this field consists in classifying those constant angle surfaces that satisfy certain geometrical properties, such as being minimal, umbilical, etcetera.

In recent years, a renewed interest for such surfaces has grown and several interesting generalizations of the concept have arisen. For instance, the ambient space $\mathbb{R}^3$ has been replaced by cartesian products or warped products, or even Lorentzian 3-manifolds. Just to mention a few of the most relevant developments in this scenario, we have the classification of surfaces in $\mathbb{R} \times \mathbb{S}^2$ and $\mathbb{R} \times \mathbb{H}^2$ making a constant angle with respect to a constant field in the first factor [6,7,9]; or non-degenerate surfaces in Lorentz–Minkowski 3-space $\mathbb{R}^3_1$ making a constant angle with a fixed non-lightlike direction [20,17]. Going one step further, notice that in the aforementioned cases the distinguished directions project to principal directions on M with non-vanishing principal curvatures. Surfaces having this property are said to have a canonical principal direction and have been extensively studied both in the Riemannian and Lorentzian settings [5,8,13].

Another suitable generalization consists in replacing constant (i.e. parallel) directions for other types of distinguished vector fields that carry geometrical relevance, such as Killing fields [22]. Closed and conformal vector fields are a natural generalization of both parallel and Killing vector fields, since they are infinitesimal generators of conformal mappings that are locally gradient fields. The presence of a closed and conformal vector field in the ambient space is a powerful tool that can be used to establish classification results, as is illustrated by the work of S. Montiel [21] and A. Barros et al. [2] on spacelike hypersurfaces of constant mean curvature. In [16] hypersurfaces in Riemannian warped products making a constant angle with respect to a closed and conformal vector field are classified. In [24] and [1] similar classification results are established for Lorentzian 3-manifolds and Riemannian 3-dimensional space-forms, respectively.

In this work we study constant angle CMC hypersurfaces immersed in semi-Riemannian space forms, thus extending some of the results obtained in [1,4,13,11,16,17,20,22,24] to the context of closed and conformal vector fields. This paper is organized as follows. In section 2 we establish the notation and main results pertaining semi-Riemannian space forms and closed and conformal vector fields. In particular, we show that in any semi-Riemannian space form, any tangent vector can be extended locally to a closed and conformal vector field, which is essentially the projection of a parallel vector field. Moreover, in a space form of non-vanishing curvature, any closed and conformal vector field can be realized in such a way. In this section we also find formulae for the (intrinsic) Laplacian of the squared norm of a closed and conformal vector field defined along a semi-Riemannian hypersurface. In section 3 we develop the concept of a constant angle hypersurface in a semi-Riemannian manifold and prove under mild assumptions that hypersurfaces that make a constant angle with respect to a closed and conformal vector field have a canonical principal direction. In section 4 we deal with two special cases of geometric significance, namely, when Z is either orthogonal or tangent to M, and provide a full description in the semi-Euclidean scenario. Finally, in section 5 we present the classification of CMC surfaces in three dimensional semi-Riemannian space forms having a constant angle with respect to a closed and conformal vector field.

2. Preliminaries

Let us denote by $\mathbb{R}^{n+2}_s$ the semi-Euclidean space given by the real vector space $\mathbb{R}^{n+2}$ endowed with the semi-Riemannian metric

$$\langle u, v\rangle = -u_1 v_1 - \ldots - u_s v_s + u_{s+1} v_{s+1} + \ldots + u_{n+2} v_{n+2},$$

and recall that a vector $u \in T_p\mathbb{R}^{n+2}_s$ is called *timelike, spacelike* or *lightlike* if $\langle u, u\rangle$ is negative, positive, or zero, respectively. Furthermore, if u is non-lightlike then $\epsilon_u = \pm 1$ will denote the sign of $\langle u, u\rangle$.

The non-degenerate hyperquadrics in $\mathbb{R}^{n+2}_s$ are totally umbilical and geodesically complete hypersurfaces with constant sectional curvature, and thus are locally isometric to a semi-Riemannian space form although not simply connected in some cases. Since our analysis is purely local, we identify these hyperquadrics with the semi-Riemannian space forms of corresponding dimension and signature.

The *pseudosphere* of radius $r > 0$ is the hypersurface defined by

$$\mathbb{S}_s^{n+1}(r) := \{x \in \mathbb{R}_s^{n+2} \mid -x_1^2 - \ldots - x_s^2 + x_{s+1}^2 + \ldots + x_{n+2}^2 = r^2\},$$

which corresponds to the semi-Riemannian space form of constant sectional curvature $c = 1/r^2$ of dimension $n+1$ and index s.

Similarly, the *pseudohyperbolic* space form of radius $r > 0$ is the hypersurface

$$\mathbb{H}_s^{n+1}(r) := \{x \in \mathbb{R}_{s+1}^{n+2} \mid -x_1^2 - \ldots - x_{s+1}^2 + x_{s+2}^2 + \ldots + x_{n+2}^2 = -r^2\},$$

with dimension $n+1$, index s and constant curvature $c = -1/r^2$.

Remark 2.1. For simplicity, we denote the space forms $\mathbb{R}_s^{n+1}$, $\mathbb{S}_s^{n+1}(r)$ and $\mathbb{H}_s^{n+1}(r)$ by $\overline{\mathbb{M}}_s^{n+1}(c)$ where $c = 0$, $c = 1/r^2$ or $c = -1/r^2$ respectively. In a similar fashion, we denote by $\mathbb{R}_\nu^{n+2}$ either of the semi-Euclidean ambient spaces $\mathbb{R}_s^{n+2}$ or $\mathbb{R}_{s+1}^{n+2}$. When $c = 0$ we take a totally geodesic immersion of $\mathbb{R}_s^{n+1}$ through the origin of $\mathbb{R}_\nu^{n+2}$.

In what follows, we will denote by D the Levi-Civita connection of the semi-Euclidean spaces $\mathbb{R}_\nu^{n+2}$ while $\overline{\nabla}$ will denote the Levi-Civita connection of $\overline{\mathbb{M}}_s^{n+1}(c)$. Lastly, let us recall that since the space forms $\overline{\mathbb{M}}_s^{n+1}(c)$ are manifolds of constant sectional curvature c, its Riemann curvature tensor satisfies

$$\overline{R}(X, Y)Z = c(\langle Y, Z\rangle X - \langle X, Z\rangle Y) \tag{1}$$

for all smooth vector fields $X, Y, Z \in \Gamma(T\overline{\mathbb{M}}_s^{n+1}(c))$.

We now define a class of vector fields that will be the key in this work.

Definition 2.2. A nonzero smooth vector field Z on a semi-Riemannian manifold $\overline{M}$ is called *closed and conformal* if there is a function $\varphi \in C^\infty(\overline{M})$ such that $\overline{\nabla}_Y Z = \varphi Y$ for every smooth vector field Y on $\overline{M}$. We call φ an *associated function* of Z.

Since closed and conformal vector fields can be thought as infinitesimal generators of conformal mappings, they are usually present in semi-Riemannian contexts where conformal symmetries play an important role. For instance, in the Lorentzian setting we have that lightlike (pre)geodesics are conformal invariants and thus closed and conformal vector fields arise naturally in scenarios like the study of plane gravitational waves [18,10].

From the definition, we immediately see that parallel vector fields are closed and conformal with $\varphi \equiv 0$. Let us point out that not all semi-Riemannian manifolds admit a non-parallel closed and conformal vector field. For instance, in the Riemannian setting no compact manifold with non-positive Ricci curvature has a non-parallel closed and conformal vector field [25], whereas in the presence of signature only a few such manifolds exist [19]. However, as the following result shows, this is not the case for semi-Riemannian space forms. Even further, any tangent vector $u \in T_p\overline{\mathbb{M}}_s^{n+1}(c)$ can be extended to a closed and conformal vector field Z in $\overline{\mathbb{M}}_s^{n+1}(c)$.

Lemma 2.3. *For every $p \in \overline{\mathbb{M}}_s^{n+1}(c)$ and every $u \in T_p\overline{\mathbb{M}}_s^{n+1}(c)$ there exists a closed and conformal vector field Z on $\overline{\mathbb{M}}_s^{n+1}(c)$ such that $Z(p) = u$. Moreover, the associated function of Z is given by*

$$\varphi(x) = -c\langle U(x), x\rangle,$$

where U is the constant vector field along $\overline{\mathbb{M}}_s^{n+1}(c)$ extending u and satisfies:

$$X \cdot \varphi = -c\langle Z, X\rangle, \qquad \langle Z(x), Z(x)\rangle = \langle u, u\rangle - \varphi^2/c,$$

for all $X \in \Gamma(\overline{\mathbb{M}}_s^{n+1}(c))$.

Proof. Using the canonical embedding of the space form $\overline{\mathbb{M}}_s^{n+1}(c)$ in a semi-Euclidean space $\mathbb{R}_\nu^{n+2}$ we can extend u to a constant vector field $U\colon \overline{\mathbb{M}}_s^{n+1}(c) \to T\mathbb{R}_\nu^{n+2}$ along $\overline{\mathbb{M}}_s^{n+1}(c)$ and thus define Z by

$$Z(x) := U(x) - c\langle U(x), x\rangle x.$$

First notice that in the case $c = 0$ the vector field $Z(x) = U(x)$ is parallel, hence closed and conformal with $\varphi \equiv 0$ and the claims follow immediately. On the other hand, if $c \neq 0$ notice that for every $x \in \overline{\mathbb{M}}_s^{n+1}(c)$ we have $\langle x, x\rangle = 1/c$ with x orthogonal to $T_x\overline{\mathbb{M}}_s^{n+1}(c)$ and in particular, $\langle p, u\rangle = 0$. These facts imply that $\langle x, Z(x)\rangle = 0$, i.e. $Z(x) \in T_x\overline{\mathbb{M}}_s^{n+1}(c)$ and $Z(p) = u$. Further, let X be a vector field in $\overline{\mathbb{M}}_s^{n+1}(c)$ and note that

$$\begin{aligned}\overline{\nabla}_X Z &= (D_X Z)^\top = (-cD_X(\langle U, x\rangle x))^\top \\ &= (-c\langle U, X\rangle x - c\langle U, x\rangle X)^\top \\ &= -c\langle U, x\rangle X.\end{aligned}$$

Thus, Z is closed and conformal with $\varphi(x) = -c\langle U(x), x\rangle$. Moreover, a straightforward calculation shows that

$$X \cdot \varphi = -c(X \cdot \langle U, x\rangle) = -c\langle U, X\rangle = -c\langle Z, X\rangle.$$

Finally, note that $\varphi\langle U, x\rangle = -\varphi^2/c = -\varphi^2\langle x, x\rangle$. Thus

$$\langle Z, Z\rangle = \langle U, U\rangle + 2\varphi\langle U, x\rangle + \varphi^2\langle x, x\rangle = \langle u, u\rangle - \varphi^2/c. \qquad \square$$

Remark 2.4. Let us observe that in Lemma 2.3, if $c > 0$ and u is timelike then Z is a timelike vector field on $\overline{\mathbb{M}}_s^{n+1}(c)$. Similarly, if $c < 0$ and u is spacelike then Z is everywhere spacelike as well.

Let us notice that Lemma 2.3 gives us a method for constructing closed and conformal vector fields on $\overline{\mathbb{M}}_s^{n+1}(c)$. We now show that essentially all closed and conformal vector fields on $\overline{\mathbb{M}}_s^{n+1}(c)$ come from such a construction. In order to do so, we first show that in semi-Riemannian space forms closed and conformal vector fields are gradient fields of its associated functions.

Lemma 2.5. *Let Z be a closed and conformal vector field on a semi-Riemannian space form $\overline{\mathbb{M}}_s^{n+1}(c)$ with associated function φ. Then $\overline{\nabla}\varphi = -cZ$.*

Proof. Since Z is closed and conformal then for all $X, Y \in \Gamma(T\overline{\mathbb{M}}_s^{n+1}(c))$ we have

$$\overline{\nabla}_X\overline{\nabla}_Y Z = \overline{\nabla}_X(\varphi Y) = (X \cdot \varphi)Y + \varphi\overline{\nabla}_X Y = \langle X, \overline{\nabla}\varphi\rangle Y + \varphi\overline{\nabla}_X Y.$$

In particular, if $X = \partial_i$, $Y = \partial_j$ are two local coordinate vector fields around $p \in \overline{\mathbb{M}}_s^{n+1}(c)$, in virtue of the above calculations and equation (1) we have two different expressions for the Riemann tensor given by:

$$\begin{aligned}\overline{R}(\partial_i, \partial_j)Z &= \langle \partial_i, \overline{\nabla}\varphi\rangle\partial_j - \langle \partial_j, \overline{\nabla}\varphi\rangle\partial_i, \\ \overline{R}(\partial_i, \partial_j)Z &= c(\langle \partial_j, Z\rangle\partial_i - \langle \partial_i, Z\rangle\partial_j).\end{aligned}$$

By comparing the above expressions we conclude that $\langle \partial_i, cZ + \overline{\nabla}\varphi \rangle = 0$ for all coordinate vectors ∂_i and thus $cZ + \overline{\nabla}\varphi = 0$. □

Corollary 2.6. *Let Z be a closed and conformal vector field on $\overline{\mathbb{M}}_s^{n+1}(c)$ with associated function φ. Then $Z(x) = U + \varphi x$, where U is a parallel vector field of the corresponding semi-Euclidean space $\mathbb{R}_\nu^{n+2}$. If $c = 0$ then φ is constant and therefore Z is either parallel or radial. If $c \neq 0$, $\varphi = -c\langle U, x\rangle$ and $\langle Z, Z\rangle = \langle U, U\rangle - \varphi^2/c$.*

Proof. Let x be the position vector field of $\mathbb{R}_\nu^{n+2}$. Observe that $\overline{\mathbb{M}}_s^{n+1}(c) \subset \mathbb{R}_\nu^{n+2}$ is totally umbilical since its second fundamental form h satisfies

$$h(X, W) = -c\langle X, W\rangle x\,,$$

for every $X, W \in \Gamma(T\overline{\mathbb{M}}_s^{n+1}(c))$. Set $U := Z - \varphi x$. We claim that U is parallel along $\overline{\mathbb{M}}_s^{n+1}(c)$. Indeed, by Lemma 2.5 we have

$$\begin{aligned} D_W U &= D_W Z - (W \cdot \varphi)x - \varphi D_W x \\ &= \overline{\nabla}_W Z + h(Z, W) + c\langle Z, W\rangle x - \varphi W \\ &= \varphi W + h(Z, W) + c\langle Z, W\rangle x - \varphi W = 0\,, \end{aligned}$$

thus proving the claim. Moreover, if $c = 0$, Lemma 2.5 shows that $\overline{\nabla}\varphi = 0$, hence $Z = U + \varphi x$ with U parallel and φ constant along $\overline{\mathbb{M}}_s^{n+1}(c)$. It immediately follows that Z is either parallel (when $\varphi \equiv 0$) or radial with center in U (when $\varphi \neq 0$). Finally when $c \neq 0$, from the relation $Z = U + \varphi x$ we deduce that

$$0 = \langle Z, x\rangle = \langle U, x\rangle + \varphi/c,$$

from which $\varphi = -c\langle U, x\rangle$ and $\langle Z, Z\rangle = \langle U, U\rangle - \varphi^2/c$. □

Another interesting feature of closed and conformal vector fields is that their norm is constant along orthogonal vector fields.

Lemma 2.7. *Let Z be a closed and conformal vector field on $\overline{\mathbb{M}}_s^{n+1}(c)$ with $|Z| \neq 0$ and $\varphi \neq 0$. Then a vector field $X \in \Gamma(\overline{\mathbb{M}}_s^{n+1}(c))$ is orthogonal to Z if and only if $X \cdot |Z| = 0$.*

Proof. Let ϵ_Z be the sign of $\langle Z, Z\rangle$. Thus, by letting X act on both sides of the equation $|Z|^2 = \epsilon_Z\langle Z, Z\rangle$, we have

$$2|Z|X \cdot |Z| = 2\epsilon_Z\langle \overline{\nabla}_X Z, Z\rangle = 2\epsilon_Z\varphi\langle X, Z\rangle = 0.$$

Thus $\langle X, Z\rangle = 0$ if and only if $X \cdot |Z| = 0$. □

Definition 2.8. We say that an immersed hypersurface $M \hookrightarrow \overline{\mathbb{M}}_s^{n+1}(c)$ is *semi-Riemannian* if its induced metric is non-degenerate. In this context, the Levi-Civita connection of M will be denoted by ∇.

Lemma 2.9. *Let $M \subset \overline{\mathbb{M}}_s^{n+1}(c)$ be a semi-Riemannian hypersurface and let Z be a closed and conformal vector field on $\overline{\mathbb{M}}_s^{n+1}$ with associated function φ. Then*

$$\nabla\varphi = -cZ^\top.$$

Proof. Let $X \in \Gamma(TM)$ be any vector field on M. Then using Lemma 2.5 we have

$$\langle \nabla\varphi, X\rangle = X \cdot \varphi = \langle \overline{\nabla}\varphi, X\rangle = \langle -cZ, X\rangle = \langle -cZ^\top, X\rangle.$$

This finishes the proof since the induced metric in M is non-degenerate. □

If M is a semi-Riemannian hypersurface of $\overline{\mathbb{M}}_s^{n+1}(c)$, we can always find a non-lightlike local unitary vector field ξ orthogonal to M. Thus, we can decompose any vector field Z of $\overline{\mathbb{M}}_s^{n+1}(c)$ in its tangent and normal components $Z^\top$ and $Z^\perp$. Moreover, using this decomposition the standard form of the Gauss and Weingarten formulae can be written as

$$\begin{aligned}\overline{\nabla}_X Y &= \nabla_X Y + \alpha(X, Y),\\ \overline{\nabla}_X N &= -A_N(X) + \nabla^\perp_X N,\end{aligned}$$

for all $X, Y \in \Gamma(TM)$, $N \in \Gamma(T^\perp M)$. See, for example [3,23]. Here A_N denotes the shape operator of N, α the second fundamental form of M and $\nabla^\perp$ the normal connection of M. Let us recall that M is said to be *totally umbilical* if there exists a smooth function λ such that $A_N(X) = \lambda X$ for all $X \in \Gamma(TM)$. If $\lambda \equiv 0$ (or equivalently, the second fundamental form is null: $\alpha = 0$) then M is *totally geodesic*. As a direct application of the Gauss–Weingarten formulae we establish the following lemmas:

Lemma 2.10. *Let $M \subset \overline{\mathbb{M}}_s^{n+1}(c)$ be a semi-Riemannian hypersurface and Z a closed and conformal vector field on $\overline{\mathbb{M}}_s^{n+1}(c)$ with associated function φ. Then*

$$\begin{aligned}\nabla_X Z^\top &= \varphi X + A_{Z^\perp}(X)\\ \alpha(X, Z^\top) &= -\nabla^\perp_X Z^\perp.\end{aligned}$$

In particular, if $T = Z^\top/|Z^\top|$ then

$$\nabla_{Z^\top} T = A_{Z^\perp}(T).$$

Proof. Since Z is closed and conformal, we apply Gauss and Weingarten formulae to find

$$\begin{aligned}\varphi X = \overline{\nabla}_X Z &= \overline{\nabla}_X Z^\top + \overline{\nabla}_X Z^\perp\\ &= \nabla_X Z^\top + \alpha(X, Z^\top) - A_{Z^\perp}(X) + \nabla^\perp_X Z^\perp.\end{aligned}$$

Comparing tangent and normal components at both sides of the equation gives the general result at once. In a similar fashion, let us notice that

$$\begin{aligned}\nabla_{Z^\top}\frac{Z^\top}{|Z^\top|} &= \frac{1}{|Z^\top|}\nabla_{Z^\top} Z^\top + \left(Z^\top \cdot \frac{1}{|Z^\top|}\right) Z^\top\\ &= \left(\frac{\varphi}{|Z^\top|} + Z^\top \cdot \frac{1}{|Z^\top|}\right) Z^\top + A_{Z^\perp}(Z^\top).\end{aligned}$$

Since $T = Z^\top/|Z^\top|$ is unitary, we have that $\nabla_{Z^\top} T$ is orthogonal to Z^T, thus finishing the proof. □

Lemma 2.11. *Let $M \subset \overline{\mathbb{M}}_s^{n+1}(c)$ be a semi-Riemannian hypersurface and Z a closed and conformal vector field on $\overline{\mathbb{M}}_s^{n+1}(c)$ with $|Z| \neq 0$. If Z is orthogonal to M then M is totally umbilical.*

Proof. Since Z is orthogonal to M then $\xi = Z/|Z|$ is a unitary normal vector field defined along M. Let $X \in \Gamma(TM)$. Then, by Lemma 2.7 we have

$$A_\xi(X) = -\overline{\nabla}_X \xi = -\left(X \cdot \frac{1}{|Z|}\right) Z - \frac{1}{|Z|}\overline{\nabla}_X Z$$
$$= -\frac{\varphi}{|Z|} X. \qquad \square$$

We now proceed to find expressions for the intrinsic Laplacians of both the squared norm of a closed and conformal vector field defined along M and the squared norm of its tangent component. This latter formula generalizes to semi-Riemannian space forms a previous technique developed by D. Fetcu and H. Rosenberg in the context of parallel vector fields in $\mathbb{H}^3 \times \mathbb{R}$ and $\mathbb{S}^3 \times \mathbb{R}$ (see p. 715 in [12]). These results will be key ingredients for the analysis of constant angle hypersurfaces that we will perform in sections 4 and 5. First, recall that the mean curvature vector can be written in an orthonormal frame $\{e_1, \ldots, e_n\}$ as

$$H = \sum_{i=1}^{n} \epsilon_i \alpha(e_i, e_i), \tag{2}$$

where α is the second fundamental form of M and ϵ_i is the sign of $\langle e_i, e_i\rangle$. We also have the following expression for the (signed) square norm of α:

$$\langle \alpha, \alpha\rangle = \sum_{i,j=1}^{n} \epsilon_i \epsilon_j \langle \alpha(e_i, e_j), \alpha(e_i, e_j)\rangle \tag{3}$$

Proposition 2.12. *Let M be a semi-Riemannian hypersurface isometrically immersed in $\overline{\mathbb{M}}_s^{n+1}(c)$ with mean curvature vector H. If Z is a closed and conformal vector field on $\overline{\mathbb{M}}_s^{n+1}(c)$ with associated function φ, then*

$$\triangle\langle Z, Z\rangle = 2(\langle \nabla\varphi, Z\rangle + \varphi\langle H, Z\rangle + \varphi^2 n). \tag{4}$$

Proof. We will prove the relation pointwise. Let $p \in M$ and $\{e_1, \ldots, e_n\}$ be a local frame in M around p such that $\nabla_{e_i} e_j|_p = 0$ for all $i,\ j$. Let $\epsilon_i = \langle e_i, e_i\rangle$. Then

$$\triangle\langle Z, Z\rangle = \sum_{i=1}^{n} \epsilon_i e_i \cdot e_i \cdot \langle Z, Z\rangle = 2\sum_{i=1}^{n} \epsilon_i e_i \cdot \langle \varphi e_i, Z\rangle$$
$$= 2\sum_{i=1}^{n} \epsilon_i[\langle (e_i \cdot \varphi) e_i, Z\rangle + \varphi(\langle \overline{\nabla}_{e_i} e_i, Z\rangle + \langle e_i, \overline{\nabla}_{e_i} Z\rangle)]$$
$$= 2\sum_{i=1}^{n} \epsilon_i[\langle (e_i \cdot \varphi) e_i, Z\rangle + \varphi(\langle \alpha(e_i, e_i), Z\rangle + \varphi\langle e_i, e_i\rangle)]$$
$$= 2(\langle \sum_{i=1}^{n} \epsilon_i (e_i \cdot \varphi) e_i, Z\rangle + \varphi\langle \sum_{i=1}^{n} \epsilon_i \alpha(e_i, e_i), Z\rangle + \varphi^2 \sum_{i=1}^{n} \epsilon_1^2)$$
$$= 2(\langle \nabla\varphi, Z\rangle + \varphi\langle H, Z\rangle + \varphi^2 n). \qquad \square$$

Proposition 2.13. *Let M be a semi-Riemannian hypersurface isometrically immersed in $\overline{\mathbb{M}}_s^{n+1}(c)$ with second fundamental form α, mean curvature vector H and let Z be a closed and conformal vector field of $\overline{\mathbb{M}}_s^{n+1}(c)$ with associated function φ. Then the tangent component $Z^\top$ of Z along M satisfies the equation*

$$\triangle\langle Z^\top, Z^\top\rangle = 2(Z^\top \cdot \langle H, Z\rangle + n\langle \nabla\varphi, Z\rangle + Ric(Z^\top, Z^\top) \\ + \epsilon_\xi |Z^\perp|^2\langle\alpha,\alpha\rangle + 2\varphi\langle H, Z\rangle + n\varphi^2).$$

Proof. Let $p \in M$ and choose a local orthonormal frame e_i around p such that $\nabla_{e_i} e_j|_p = 0$. Then

$$\begin{aligned}\triangle\langle Z^\top, Z^\top\rangle &= \sum_{i=1}^n \epsilon_i e_i \cdot e_i \cdot \langle Z^\top, Z^\top\rangle \\ &= 2\left[\sum_{i=1}^n \epsilon_i \langle \nabla_{e_i}\nabla_{e_i} Z^\top, Z^\top\rangle + \sum_{i=1}^n \epsilon_i\langle \nabla_{e_i} Z^\top, \nabla_{e_i} Z^\top\rangle\right]. \end{aligned} \tag{5}$$

In order to find a closed expression for the second sum we notice that

$$\langle \nabla_{e_i} Z^\top, \nabla_{e_i} Z^\top\rangle = \langle A_{Z^\perp} e_i, A_{Z^\perp} e_i\rangle + 2\varphi\langle A_{Z^\perp} e_i, e_i\rangle + \varphi^2\langle e_i, e_i\rangle.$$

Thus

$$\sum_{i=1}^n \epsilon_i\langle \nabla_{e_i} Z^\top, \nabla_{e_i} Z^\top\rangle = |Z^\perp|^2\langle\alpha,\alpha\rangle\epsilon_\xi + 2\varphi\langle H, Z\rangle + n\varphi^2. \tag{6}$$

We now deal with the first sum. Since $A_{Z^\top}$ is symmetric, so it is $\nabla_{e_i} A_{Z^\top}$. Further, since $\nabla_{e_i} e_j|_p = 0$ we have $(\nabla_{e_i} A_{Z^\perp})e_i = \nabla_{e_i}(A_{Z^\perp} e_i)$. Hence

$$\begin{aligned}\sum_{i=1}^n \epsilon_i\langle \nabla_{e_i} A_{Z^\perp} e_i, Z^\top\rangle &= \sum_{i=1}^n \epsilon_i\langle (\nabla_{e_i} A_{Z^\perp})e_i, Z^\top\rangle \\ &= \sum_{i=1}^n \epsilon_i\langle e_i, (\nabla_{e_i} A_{Z^\perp})Z^\top\rangle \\ &= \sum_{i=1}^n \epsilon_i\langle e_i, \nabla_{e_i}(A_{Z^\perp} Z^\top)\rangle - \sum_{i=1}^n \epsilon_i\langle e_i, A_{Z^\perp}(\nabla_{e_i} Z^\top)\rangle. \end{aligned}$$

Now we proceed to compute each of these two sums separately. First notice that

$$\begin{aligned}\sum_{i=1}^n \epsilon_i\langle e_i, \nabla_{e_i}(A_{Z^\perp} Z^\top)\rangle &= \sum_{1=i}^n \epsilon_i\langle e_i, \nabla_{e_i}(\nabla_{Z^\top} Z^\top - \varphi Z^\top)\rangle \\ &= \sum_{i=1}^n \epsilon_i\langle e_i, \nabla_{e_i}\nabla_{Z^\top} Z^\top\rangle - \sum_{i=1}^n \epsilon_i\langle e_i, e_i \cdot \varphi Z^\top\rangle - \sum_{i=1}^n \epsilon_i\langle e_i, \varphi\nabla_{e_i} Z^\top\rangle \\ &= \sum_{i=1}^n \epsilon_i\langle e_i, \nabla_{e_i}\nabla_{Z^\top} Z^\top\rangle - \langle\nabla\varphi, Z\rangle - \varphi \operatorname{div} Z^\top. \end{aligned}$$

On the other hand, since $\nabla_{\nabla_{Z^\top} e_i} Z^\top|_p = 0$ we have

$$\begin{aligned}\sum_{i=1}^n \epsilon_i\langle e_i, A_{Z^\perp}(\nabla_{e_i} Z^\top)\rangle &= \sum_{i=1}^n \epsilon_i\langle e_i, \nabla_{\nabla_{e_i} Z^\top} Z^\top - \varphi\nabla_{e_i} Z^\top\rangle \\ &= \sum_{i=1}^n \epsilon_i\langle e_i, -\nabla_{[Z^\top, e_i]} Z^\top\rangle - \varphi \operatorname{div} Z^\top. \end{aligned}$$

Thus we have

$$\begin{aligned}
\sum_{i=1}^{n} \epsilon_i \langle \nabla_{e_i} A_{Z^\perp} e_i, Z^\top \rangle &= \sum_{i=1}^{n} \epsilon_i \langle e_i, \nabla_{e_i} \nabla_{Z^\top} Z^\top + \nabla_{[Z^\top, e_i]} Z^\top \rangle - \langle \nabla\varphi, Z \rangle \\
&= \sum_{i=1}^{n} \epsilon_i \langle e_i, \nabla_{Z^\top} \nabla_{e_i} Z^\top - R_{Z^\top e_i} Z^\top \rangle - \langle \nabla\varphi, Z \rangle \\
&= \sum_{i=1}^{n} \epsilon_i \langle e_i, \nabla_{Z^\top} (A_{Z^\perp} e_i + \varphi e_i) \rangle + Ric(Z^\top, Z^\top) - \langle \nabla\varphi, Z \rangle \\
&= \sum_{i=1}^{n} \epsilon_i \langle \nabla_{Z^\top} A_{Z^\perp} e_i, e_i \rangle + \sum_{i=1}^{n} \epsilon_i \langle Z^\top \cdot \varphi e_i, e_i \rangle + Ric(Z^\top, Z^\top) - \langle \nabla\varphi, Z \rangle \\
&= \sum_{i=1}^{n} \epsilon_i \langle \nabla_{Z^\top} A_{Z^\perp} e_i, e_i \rangle + (n-1) \langle \nabla\varphi, Z \rangle + Ric(Z^\top, Z^\top) \\
&= \sum_{i=1}^{n} \epsilon_i (Z^\top \cdot \langle A_{Z^\perp} e_i, e_i \rangle - \langle A_{Z^\perp} e_i, \nabla_{Z^\perp} e_i \rangle) + (n-1) \langle \nabla\varphi, Z \rangle + Ric(Z^\top, Z^\top) \\
&= Z^\top \cdot \langle H, Z \rangle + (n-1) \langle \nabla\varphi, Z \rangle + Ric(Z^\top, Z^\top).
\end{aligned}$$

Notice that we can find now the first sum as follows:

$$\begin{aligned}
\sum_{i=1}^{n} \epsilon_i \langle \nabla_{e_i} \nabla_{e_i} Z^\top, Z^\top \rangle &= \sum_{i=1}^{n} \epsilon_i \langle \nabla_{e_i} (A_{Z^\perp} e_i + \varphi e_i), Z^\top \rangle \\
&= \sum_{i=1}^{n} \epsilon_i \langle \nabla_{e_i} A_{Z^\perp} e_i, Z^\top \rangle + \sum_{i=1}^{n} \epsilon_i \langle e_i \cdot \varphi e_i, Z^\top \rangle + \sum_{i=1}^{n} \epsilon_i \langle \varphi \nabla_{e_i} e_i, Z^\top \rangle \\
&= Z^\top \cdot \langle H, Z \rangle + n \langle \nabla\varphi, Z \rangle + Ric(Z^\top, Z^\top). \qquad (7)
\end{aligned}$$

Finally, by substituting equations (6) and (7) in (5) we get the desired formula for the Laplacian. □

3. Constant angle hypersurfaces

In the Riemannian setting, the angle between two hypersurfaces can be defined as the angle spanned by their normal vectors. In a similar fashion, the angle between a hypersurface M and a given vector field X along it can be measured in terms of the angle spanned by the unit normal vector field ξ and X, and thus, if the function $\langle X/|X|, \xi \rangle$ is constant on M we say M is a constant angle hypersurface with respect to X. This notion can be easily generalized to the Lorentzian setting provided M is semi-Riemannian and X is non-lightlike [13,24]. The following definition is a further generalization of this concept to semi-Riemannian manifolds of any signature.

Definition 3.1. Let M be a semi-Riemannian hypersurface in $\overline{\mathbb{M}}_s^{n+1}(c)$ and ξ a normal vector field to M in a neighborhood V of $p \in M$ with $\langle \xi, \xi \rangle^2 = 1$. Let Z be a closed and conformal vector field on $\overline{\mathbb{M}}_s^{n+1}(c)$ that does not vanish along M. We say that M is a *constant angle hypersurface* with respect to Z if the product $\langle Z/|Z|, \xi \rangle$ is constant along $V \subset M$.

In what follows, ξ will denote a local unitary vector field orthogonal to a semi-Riemannian hypersurface M isometrically immersed in $\overline{\mathbb{M}}_s^{n+1}(c)$ and Z will denote a closed and conformal vector field on $\overline{\mathbb{M}}_s^{n+1}(c)$ with associated function φ that does not vanish along M.

Remark 3.2. Let $Z = Z^\top + Z^\perp$, where $Z^\top \in TM$ and $Z^\perp \in T^\perp M$ are its tangent and normal components, respectively. Since our analysis will be local, we can assume that Z, $Z^\top$ and $Z^\perp$ have a constant causality, and we further assume none of them is lightlike. Notice that in particular none of them vanishes. That is, we will assume that

$$\epsilon_Z := \frac{\langle Z, Z\rangle}{|Z|^2},\ \epsilon_T := \frac{\langle Z^\top, Z^\top\rangle}{|Z^\top|^2} \text{ and } \epsilon_\xi := \frac{\langle Z^\perp, Z^\perp\rangle}{|Z^\perp|^2}$$

are constant nonzero locally. In particular, we have to deal with the cases in which Z is orthogonal to M (i.e. $Z^\top = 0$) or tangent to M (i.e. $Z^\perp = 0$) separately.

Lemma 3.3. *Let $Z = Z^\top + Z^\perp$ be the decomposition of Z in its tangent and normal parts along M. Then the following are equivalent:*

1. *M is a constant angle hypersurface with respect to Z.*
2. *$\langle Z/|Z|, Z^\top/|Z^\top|\rangle$ is constant.*
3. *$\langle Z^\top/|Z|, Z^\top/|Z|\rangle$ is constant.*
4. *$\langle Z^\perp/|Z|, Z^\perp/|Z|\rangle$ is constant.*

Moreover, we have the following relations

$$\begin{aligned}\langle Z^\perp, Z^\perp\rangle &= \epsilon_\xi \langle Z, \xi\rangle^2,\\ \langle Z^\top, Z^\top\rangle &= \frac{\epsilon_T \lambda^2}{\mu^2}\langle Z, \xi\rangle^2,\\ \langle Z, Z\rangle &= \frac{\epsilon_Z}{\mu^2}\langle Z, \xi\rangle^2,\end{aligned}$$

where λ and μ are such that $\dfrac{Z}{|Z|} = \lambda T + \mu\xi$, with $T = \dfrac{Z^\top}{|Z^\top|}$.

Proof. Let us observe that $Z^\top = |Z|\lambda T$ and $Z^\perp = |Z|\mu\xi$. In virtue of the relation

$$\epsilon_Z = \epsilon_T \lambda^2 + \epsilon_\xi \mu^2 \tag{8}$$

it follows that λ is constant if and only if μ is constant. Now, let us observe that

$$\begin{aligned}\langle Z/|Z|, Z^\top/|Z^\top|\rangle &= \langle Z^\top/|Z|, T\rangle = \langle \lambda T, T\rangle = \epsilon_T \lambda,\\ \langle Z^\top/|Z|, Z^\top/|Z|\rangle &= \langle \lambda T, \lambda T\rangle = \epsilon_T \lambda^2,\\ \langle Z^\perp/|Z|, Z^\perp/|Z|\rangle &= \langle \mu\xi, \mu\xi\rangle = \epsilon_\xi \mu^2.\end{aligned}$$

Moreover, by definition, M is a constant angle hypersurface with respect to Z if and only if the function $\langle Z/|Z|, \xi\rangle = \epsilon_\xi \mu$ is constant. The result readily follows. □

We continue our analysis of constant angle hypersurfaces showing that they belong to the class of hypersurfaces with a canonical principal direction. We need first an easy calculation.

Lemma 3.4. *Let $M \subset \overline{\mathbb{M}}_s^{n+1}(c)$ be a constant angle hypersurface with respect to a closed and conformal vector field Z with associated function φ. Then*

$$\nabla|Z| = \frac{\epsilon_Z \varphi}{|Z|} Z^\top$$

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal frame defined in a neighborhood of $p \in M$. Thus

$$\nabla\langle Z, Z\rangle = \sum_{i=1}^{n} \epsilon_i (e_i \cdot \langle Z, Z\rangle) e_i = 2\varphi \sum_{i=1}^{n} \epsilon_i \langle e_i, Z\rangle e_i = 2\varphi Z^\top.$$

On the other hand, $\nabla|Z|^2 = 2|Z|\nabla|Z|$. Thus

$$\nabla|Z| = \frac{\nabla \epsilon_Z \langle Z, Z\rangle}{2|Z|} = \frac{\epsilon_Z \varphi}{|Z|} Z^\top. \qquad \square$$

Proposition 3.5. *Let $M \subset \overline{\mathbb{M}}_s^{n+1}(c)$ be a constant angle hypersurface with respect to a closed and conformal vector field Z with associated function φ. Then*

$$A_\xi(Z^\top) = kZ^\top, \quad \textit{where} \quad k = -\frac{\epsilon_Z \varphi \langle Z, \xi\rangle}{|Z|^2}. \tag{9}$$

That is, $Z^\top$ is a principal direction of M with principal curvature k given by (9). Moreover, the integral curves of $T = Z^\top/|Z^\top|$ are geodesics of M.

Proof. As before, we work at $p \in M$. Let $\{e_1, \dots, e_n\}$ be an orthonormal frame at p such that $\nabla_{e_i} e_j|_p = 0$. Since Z is closed and conformal, $\langle \overline{\nabla}_{e_i} Z, \xi\rangle = \langle \varphi e_i, \xi\rangle = 0$. Thus

$$\begin{aligned}
\nabla\langle Z, \xi\rangle &= \sum_{i=1}^{n} \epsilon_i (e_i \cdot \langle Z, \xi\rangle) e_i = \sum_{i=1}^{n} \epsilon_i \langle Z, -A_\xi e_i\rangle e_i \\
&= \sum_{i=1}^{n} \epsilon_i \langle Z^\top, -A_\xi e_i\rangle e_i = \sum_{i=1}^{n} \epsilon_i \langle e_i, -A_\xi(Z^\top)\rangle e_i \\
&= -A_\xi(Z^\top).
\end{aligned}$$

A similar computation yields

$$\nabla\langle Z, Z\rangle = \sum_{i=1}^{n} \epsilon_i (e_i \cdot \langle Z, Z\rangle) e_i = 2\varphi \sum_{i=1}^{n} \epsilon_i \langle e_i, Z\rangle e_i = 2\varphi Z^\top.$$

On the other hand, $\nabla|Z|^2 = 2|Z|\nabla|Z|$ implies that

$$\nabla|Z| = \frac{\nabla \epsilon_Z \langle Z, Z\rangle}{2|Z|} = \frac{\epsilon_Z \varphi}{|Z|} Z^\top.$$

Then

$$\nabla \frac{1}{|Z|} = -\frac{\epsilon_Z \varphi}{|Z|^3} Z^\top.$$

Now, since M has constant angle with respect to Z we have

$$0 = \nabla \frac{\langle Z, \xi\rangle}{|Z|} = \langle Z, \xi\rangle \nabla \frac{1}{|Z|} + \frac{1}{|Z|} \nabla\langle Z, \xi\rangle$$

and hence

$$A_\xi(Z^\top) = -\nabla\langle Z,\xi\rangle = |Z|\langle Z,\xi\rangle\nabla\frac{1}{|Z|} = -\frac{\epsilon_Z\varphi\langle Z,\xi\rangle}{|Z|^2}Z^\top.$$

As a consequence, by Lemma 2.10 we have that $\nabla_T T$ is both a multiple of $Z^\top$ and orthogonal to it. Hence $\nabla_T T = 0$, which means that the integral lines of T are geodesics. □

Remark 3.6. Notice that the result in Proposition 3.5 is equivalent to

$$\alpha(Z^\top, Z^\top) = -\frac{\epsilon_\xi\epsilon_Z\epsilon_T\varphi|Z^\top|^2\langle Z,\xi\rangle}{|Z|^2}\xi.$$

Corollary 3.7. *Let M be a hypersurface isometrically immersed in $\overline{\mathbb{M}}_s^{n+1}(c)$ with mean curvature vector H and let Z be a closed and conformal vector field with associated function φ. If M has constant angle with respect to Z then the Ricci curvature of M in the direction $Z^\top$ is given by*

$$Ric(Z^\top, Z^\top) = \epsilon_T|Z^\top|^2\left((n-1)c - \frac{\epsilon_\xi\epsilon_Z\varphi\langle H,\xi\rangle\langle Z,\xi\rangle}{|Z|^2} - \frac{\epsilon_\xi\varphi^2\langle Z,\xi\rangle^2}{|Z|^4}\right).$$

In particular, if M is a two dimensional surface then its curvature is

$$K = c - \frac{\epsilon_\xi\epsilon_Z\varphi\langle H,\xi\rangle\langle Z,\xi\rangle}{|Z|^2} - \frac{\epsilon_\xi\varphi^2\langle Z,\xi\rangle^2}{|Z|^4}.$$

Proof. By Proposition 2.3 in [3] page 35,

$$\begin{aligned} Ric(Z^\top, Z^\top) &= (n-1)\langle Z^\top, Z^\top\rangle c + \langle H, \alpha(Z^\top, Z^\top)\rangle \\ &\quad - \sum_{i=1}^{n}\epsilon_i\langle\alpha(Z^\top, e_i), \alpha(Z^\top, e_i)\rangle, \end{aligned}$$

where $\{e_1,\dots,e_n\}$ is an orthonormal frame of M. We can choose e_1 to be $Z^\top/|Z^\top|$. Then, by Proposition 3.5, we have $\alpha(Z^\top, e_j) = 0$ for every $j \geq 2$ and

$$\begin{aligned} \langle H, \alpha(Z^\top, Z^\top)\rangle &= -\frac{\epsilon_\xi\epsilon_Z\epsilon_T\varphi|Z^\top|^2\langle Z,\xi\rangle\langle H,\xi\rangle}{|Z|^2} \\ \langle\alpha(Z^\top, Z^\top), \alpha(Z^\top, Z^\top)\rangle &= \frac{\epsilon_\xi\varphi^2|Z^\top|^4\langle Z,\xi\rangle^2}{|Z|^4}. \end{aligned}$$

The formulae follow from a straightforward substitution. □

The next result characterizes the hypersurfaces in a semi-Euclidean space with zero mean curvature and making a constant angle with respect to a radial vector field in the ambient space.

Corollary 3.8. *Let $M \subset \mathbb{R}_s^{n+1}$ be a constant angle hypersurface with respect to a radial vector field Z in $\mathbb{R}_s^{n+1}$. If M has zero mean curvature then $Ric(Z^\top, Z^\top) = -\epsilon_\xi\epsilon_T\lambda^2\mu^2\varphi^2$ is constant. In particular, if this constant is zero then either Z is tangent or orthogonal to M.*

Proof. Since Z is radial we have that $\varphi \neq 0$ is constant. Since $H = 0$ and $c = 0$ by hypothesis, Corollary 3.7 yields

$$Ric(Z^\top, Z^\top) = -\frac{\epsilon_\xi \epsilon_T |Z^\top|^2 \varphi^2 \langle Z, \xi\rangle^2}{|Z|^4}.$$

Thus $Ric(Z^\top, Z^\top) = 0$ if and only if Z is either tangent (i.e. $\langle Z, \xi\rangle = 0$) or orthogonal (i.e. $Z^\top = 0$) to M. Otherwise, we can use the relations in Lemma 3.3 to conclude that $Ric(Z^\top, Z^\top) = -\epsilon_\xi \epsilon_T \lambda^2 \mu^2 \varphi^2$ is a non-zero constant. □

Proposition 3.9. *Let $M \subset \overline{\mathbb{M}}_s^{n+1}(c)$ be a constant angle hypersurface with respect to a closed and conformal vector field Z with associated function φ and let H be the mean curvature vector of M. Then*

$$\triangle\langle Z^\top, Z^\top\rangle = \frac{2\epsilon_T \epsilon_Z |Z^\top|^2}{|Z|^2}\left(\langle \nabla\varphi, Z\rangle + \varphi\langle H, Z\rangle + \varphi^2 n\right).$$

Proof. Since M has constant angle with respect to Z, by Lemma 3.3 we have that λ is constant. Thus we take the Laplacian in both sides of $\langle Z^\top, Z^\top\rangle = \epsilon_T \epsilon_Z \lambda^2 \langle Z, Z\rangle$ to obtain

$$\begin{aligned}\triangle\langle Z^\top, Z^\top\rangle &= \epsilon_T \epsilon_Z \lambda^2 \triangle\langle Z, Z\rangle \\ &= 2\epsilon_T \epsilon_Z \lambda^2 (\langle \nabla\varphi, Z\rangle + \varphi\langle H, Z\rangle + \varphi^2 n)\end{aligned}$$

and the result follows. □

4. CMC constant angle hypersurfaces

In this section we use the results proven so far in the analysis of constant mean curvature (CMC) hypersurfaces having a constant angle with respect to a closed and conformal vector field. As a first step, we derive a formula that generalizes J. Simons' formula for the intrinsic Laplacian of the normal component of a parallel vector field to our context (refer to [26], p. 89).

4.1. Simons type formula and applications

Theorem 4.1. *Let M be a semi-Riemannian hypersurface isometrically immersed in $\overline{\mathbb{M}}_s^{n+1}(c)$ with CMC and let ξ be a local unitary vector field orthogonal to M. If Z is a closed and conformal vector field on $\overline{\mathbb{M}}_s^{n+1}(c)$ with associated function φ, then*

$$\triangle\langle Z, \xi\rangle + \langle \alpha, \alpha\rangle\langle Z, \xi\rangle + \varphi\langle H, \xi\rangle = 0, \tag{10}$$

where α is the second fundamental form of the immersion and H is the mean curvature vector of M.

Proof. As we have done before, we consider a local frame in M around p such that $\nabla_{e_i} e_{j_{|p}} = 0$. Then by Equation (2) the Laplacian is given by

$$\begin{aligned}\triangle\langle Z, \xi\rangle &= \sum_{i=1}^n \epsilon_i e_i \cdot e_i \cdot \langle Z, \xi\rangle \\ &= \sum_{i=1}^n \epsilon_i e_i \cdot \langle Z, \overline{\nabla}_{e_i}\xi\rangle = -\sum_{i=1}^n \epsilon_i e_i \cdot \langle Z, A_\xi(e_i)\rangle \\ &= -\varphi \sum_{i=1}^n \epsilon_i \langle e_i, A_\xi(e_i)\rangle - \sum_{i=1}^n \epsilon_i \langle Z, \overline{\nabla}_{e_i} A_\xi(e_i)\rangle \\ &= -\varphi\langle H, \xi\rangle - \sum_{i=1}^n \epsilon_i \langle Z, \nabla_{e_i} A_\xi(e_i)\rangle - \sum_{i=1}^n \epsilon_i \langle Z, \alpha(e_i, A_\xi(e_i))\rangle. \end{aligned} \tag{11}$$

We now compute the last term in the right hand side of Equation (11) taking into account Equation (3):

$$\begin{aligned}\sum_{i=1}^{n}\epsilon_i\langle Z,\alpha(e_i,A_\xi(e_i))\rangle &= \sum_{i=1}^{n}\epsilon_i\langle Z,\alpha(e_i,\sum_{j=1}^{n}\epsilon_j\langle A_\xi(e_i)),e_j\rangle e_j)\rangle\\ &= \sum_{i=1}^{n}\epsilon_i\epsilon_j\langle Z,\alpha(e_i,e_j)\rangle\langle\alpha(e_i,e_j),\xi\rangle\\ &= \sum_{i=1}^{n}\epsilon_\xi\epsilon_i\epsilon_j\langle\alpha(e_i,e_j),\xi\rangle^2\langle Z,\xi\rangle\\ &= \sum_{i=1}^{n}\epsilon_i\epsilon_j\langle\alpha(e_i,e_j),\alpha(e_i,e_j)\rangle\langle Z,\xi\rangle\\ &= \langle\alpha,\alpha\rangle\langle Z,\xi\rangle.\end{aligned}$$

Finally, let us show that the remaining term in the right hand side of equation (11) vanishes. First we consider

$$\begin{aligned}\nabla_{e_i}A_\xi(e_i) &= \sum_{j=1}^{n}\nabla_{e_i}(\epsilon_j\langle A_\xi(e_i),e_j\rangle e_j) = \sum_{j=1}^{n}\epsilon_j(e_i\cdot\langle\alpha(e_i,e_j),\xi\rangle)e_j\\ &= \sum_{j=1}^{n}\epsilon_j(e_i\cdot\langle\overline{\nabla}_{e_j}e_i,\xi\rangle)e_j = \sum_{j=1}^{n}\epsilon_j\langle\overline{\nabla}_{e_i}\overline{\nabla}_{e_j}e_i,\xi\rangle e_j.\end{aligned}$$

By the curvature formula (1) we have $\langle\overline{\nabla}_{e_i}\overline{\nabla}e_je_k,\xi\rangle = \langle\overline{\nabla}_{e_j}\overline{\nabla}e_ie_k,\xi\rangle$ and thus

$$\begin{aligned}\nabla_{e_i}A_\xi(e_i) &= \sum_{j=1}^{n}\epsilon_j\langle\overline{\nabla}_{e_j}\overline{\nabla}_{e_i}e_i,\xi\rangle e_j\\ &= \sum_{j=1}^{n}\epsilon_j\langle\overline{\nabla}_{e_j}\nabla_{e_i}e_i,\xi\rangle e_j + \sum_{j=1}^{n}\epsilon_j\langle\overline{\nabla}_{e_j}\alpha(e_i,e_i),\xi\rangle e_j\\ &= \sum_{j=1}^{n}\epsilon_j\langle\alpha(e_j,\nabla_{e_i}e_i),\xi\rangle e_j + \sum_{j=1}^{n}\epsilon_j\langle\nabla^{\perp}_{e_j}\alpha(e_i,e_i),\xi\rangle e_j\\ &= \sum_{j=1}^{n}\epsilon_j\langle\nabla^{\perp}_{e_j}\alpha(e_i,e_i),\xi\rangle e_j.\end{aligned}$$

Finally, since M has constant mean curvature, we have

$$\begin{aligned}\sum_{i=1}^{n}\epsilon_i\nabla_{e_i}A_\xi(e_i) &= \sum_{j=1}^{n}\epsilon_j\langle\nabla^{\perp}_{e_j}\sum_{i=1}^{n}\epsilon_i\alpha(e_i,e_i),\xi\rangle e_j\\ &= \sum_{j=1}^{n}\epsilon_j\langle\nabla^{\perp}_{e_j}H,\xi\rangle e_j = 0,\end{aligned}$$

and the result follows. □

As a first application of Theorem 4.1 we have the following result:

Corollary 4.2. *Let M be a semi-Riemannian hypersurface immersed in a semi-Euclidean space making a constant angle with respect to a parallel vector field Z. Then $Ric(Z^\top, Z^\top) = 0$ and the integral curves of $Z^\top$ are straight line segments in the ambient, i.e. M is ruled. Moreover, if Z is not tangent to M then M has constant mean curvature in the direction $Z^\top$ if and only if $\langle \alpha, \alpha \rangle = 0$.*

Proof. Since Z is parallel and M has constant angle, we have that $\varphi \equiv 0$ and $\langle Z, \xi \rangle$ is constant. We have $Ric(Z^\top, Z^\top) = 0$ as a consequence of Corollary 3.7. Further, Remark 3.6 implies that $\alpha(Z^\top, Z^\top) = 0$. Moreover, Lemma 2.10 coupled with Proposition 2.6 implies that $\nabla_{Z^\top} Z^\top = 0$, which in turns yields by Gauss formula

$$D_{Z^\top} Z^\top = \nabla_{Z^\top} Z^\top + \alpha(Z^\top, Z^\top) = 0,$$

so the integral curves of $Z^\top$ are straight lines. Now suppose that Z is not tangent to M, or equivalently that $\langle Z, \xi \rangle \neq 0$. Notice that by Proposition 3.9 we have that $\Delta \langle Z^\top, Z^\top \rangle = 0$. Hence Theorem 2.13 implies the relation

$$\begin{aligned} 0 &= Z^\top \cdot \langle H, Z \rangle + \epsilon_\xi |Z^\perp|^2 \langle \alpha, \alpha \rangle \\ &= \epsilon_\xi \langle Z, \xi \rangle Z^\top \cdot \langle H, \xi \rangle + \epsilon_\xi |Z^\perp|^2 \langle \alpha, \alpha \rangle. \end{aligned}$$

Thus

$$Z^\top \cdot \langle H, \xi \rangle = -\langle Z, \xi \rangle \langle \alpha, \alpha \rangle$$

and the proof is complete. □

Remark 4.3. As opposed to the Riemannian scenario, a hypersurface in a semi-Euclidean space with $\langle \alpha, \alpha \rangle = 0$ is not necessarily totally geodesic. This feature has its roots in the fact that a self-adjoint operator respect to a non-positive definite product might not be diagonalizable. However, when the shape operator of M is diagonalizable (for instance, when M is spacelike) then the condition $\langle \alpha, \alpha \rangle = 0$ guarantees that M is totally geodesic.

The next Corollary generalizes known facts in the Riemannian case but with weaker hypothesis.

Corollary 4.4. *Let M be a semi-Riemannian hypersurface immersed in a semi-Euclidean space making a constant angle with respect to a parallel vector field Z. Let us assume that Z is not tangent to M. If M has constant mean curvature in the direction $Z^\top$ then any of the two following conditions imply that M is an open part of a hyperplane:*

1. *Either M is spacelike or A_ξ is diagonalizable.*
2. *The semi-Euclidean ambient is the Minkowski space, M is timelike and Z^T is timelike.*

Proof. By Corollary 4.2, $\langle \alpha, \alpha \rangle = 0$. In virtue of Remark 4.3, we deduce that M is totally geodesic in case (1). On the other hand, if (2) holds, then since M and $Z^\top$ are timelike, then the slices $M \cap \Pi$ of M with hyperplanes Π orthogonal to Z are spacelike. Then the shape operator A_ξ of M is diagonalizable because $Z^\top$ is a principal direction with zero principal curvature. The shape operator admits an orthogonal decomposition in the direction $Z^\top$ and the directions tangent to $M \cap \Pi$ and the result follows from case (1). □

Lemma 4.5. *Let $M \subset \overline{\mathbb{M}}_s^{n+1}(c)$ be a constant angle hypersurface with respect to a closed and conformal vector field Z. If M has CMC then*

$$Z^\top \cdot \langle H, Z\rangle = \epsilon_\xi \epsilon_T \epsilon_Z \lambda^2 \varphi \langle H, \xi\rangle \langle Z, \xi\rangle.$$

Proof. First notice that $\langle H, Z\rangle = \langle H, Z^\perp\rangle = \epsilon_\xi \langle H, \xi\rangle\langle Z, \xi\rangle$. By hypothesis, $\langle H, \xi\rangle$ is constant. Thus

$$\begin{aligned} Z^\top \cdot \langle H, Z\rangle &= \epsilon_\xi \langle H, \xi\rangle Z^\top \cdot \langle Z, \xi\rangle \\ &= \epsilon_\xi \langle H, \xi\rangle \varphi \langle Z^\top, \xi\rangle + \epsilon_\xi \langle H, \xi\rangle \langle Z, \overline{\nabla}_{Z^\top}\xi\rangle \\ &= -\epsilon_\xi \langle H, \xi\rangle \langle Z^\top, A_\xi(Z^\top)\rangle \end{aligned}$$

and the result follows from Lemmas 3.5 and 3.3. □

4.2. Two special cases: Z orthogonal or tangent to M

We now proceed to briefly discuss the special cases that were not taken into account in the derivation of Lemma 3.3. In this subsection we will assume that Z is either tangent or orthogonal to M.

The orthogonal case follows immediately from Lemma 2.11.

Corollary 4.6. *Let M be a CMC hypersurface isometrically immersed in $\overline{\mathbb{M}}_s^{n+1}(c)$. If Z is a closed and conformal vector field orthogonal to M, then M is totally umbilical.*

We now analyze in detail the tangent case.

Corollary 4.7. *Let M be a CMC hypersurface isometrically immersed in $\overline{\mathbb{M}}_s^{n+1}(c)$. If Z is tangent to M then it has constant zero Gauss–Kronecker curvature and either M has zero mean curvature or $c = 0$, i.e. $\overline{\mathbb{M}}_s^{n+1}(c)$ is a semi-Euclidean space $\mathbb{R}_s^{n+1}$. Moreover, the Ricci curvature of M in the direction $Z^\top$ is given by*

$$Ric(Z^\top, Z^\top) = (n-1)\langle Z^\top, Z^\top\rangle c. \tag{12}$$

In particular, if $\dim M = 2$ *then either M is totally geodesic or $c = 0$.*

Proof. Since Z is tangent to M (i.e. $Z = Z^\top$), we deduce that $\langle Z, \xi\rangle = 0$ and in particular M has constant angle. By Proposition 3.5, we conclude that M satisfies $A_\xi(Z^\top) = 0$, i.e. zero is a principal curvature of M. This proves that M has zero Gauss–Kronecker curvature. Moreover, by Theorem 4.1, $\varphi\langle H, \xi\rangle = 0$ along M. Let us observe that by hypothesis $\langle H, \xi\rangle$ is constant, then either $\langle H, \xi\rangle \equiv 0$ or $\langle H, \xi\rangle \neq 0$. In the former case we obtain $H = 0$, i.e. M has zero mean curvature. In the latter, we deduce that $\varphi \equiv 0$ along M and this implies by Lemma 2.9 that $c = 0$. Further, by Corollary 3.7, the hypothesis $\langle Z, \xi\rangle = 0$ implies that

$$Ric(Z^\top, Z^\top) = \epsilon_T |Z^\top|^2 (n-1)c = (n-1)\langle Z^\top, Z^\top\rangle c.$$

Finally, let us consider $\dim M = 2$ and $c \neq 0$. Let $\{T, W\}$ be a local orthonormal frame in M that extends $T = Z/|Z|$. Thus

$$\epsilon_T \alpha(T, T) + \epsilon_W \alpha(W, W) = H = 0.$$

Let us recall that, by Proposition 3.5, we have the condition $\alpha(T, X) = 0$ for every $X \in \Gamma(TM)$. Thus, $\alpha(T, T) = 0 = \alpha(T, W)$ and therefore $\alpha(W, W) = 0$, which shows that $\alpha = 0$. □

We now describe in detail the case $c = 0$ when Z is either orthogonal or tangent to M. The following Lemma – whose proof is a straightforward adaptation of the one given in Y. Xin (see p. 64 in [27]) for the Riemannian case – will be key for our analysis.

Let Σ be a hypersurface in the hyperquadric either $Q = \mathbb{S}^{n+1}_s(1)$ or $Q = \mathbb{H}^{n+1}_{s-1}(1)$ of $\mathbb{R}^{n+2}_s$. If $\epsilon > 0$, we define the hypercone $C\Sigma_\epsilon$ by

$$C\Sigma_\epsilon = \{\lambda x \in \mathbb{R}^{n+2}_s \mid x \in \Sigma, \lambda \in (\epsilon, +\infty)\}.$$

Lemma 4.8. *Let Σ be a hypersurface in a hyperquadric Q of $\mathbb{R}^{n+2}_s$. If the hypercone $C\Sigma_\epsilon$ over Σ has CMC in $\mathbb{R}^{n+2}_s$ then Σ has zero mean curvature in the hyperquadric Q and therefore $C\Sigma_\epsilon$ has zero mean curvature in the semi-Euclidean ambient.*

Let us recall that, by Corollary 2.6, a non-zero closed and conformal vector field Z in a semi-Euclidean space $\mathbb{R}^{n+2}_s$ is either constant $Z(x) = U$ for some $U \in \mathbb{R}^{n+2}_s$ or radial, i.e. $Z(x) = U + ax$ for some $a \neq 0 \in \mathbb{R}$ and $U \in \mathbb{R}^{n+2}_s$.

Corollary 4.9. *Let $M \subset \mathbb{R}^{n+2}_s$ be a constant angle hypersurface with respect to a closed and conformal vector field $Z \in \Gamma(\mathbb{R}^{n+2}_s)$. If M has CMC then M is an open part of either a hyperquadric, a hyperplane, a cylinder over a CMC hypersurface in the hyperplane orthogonal to Z or a hypercone with zero mean curvature over a hypersurface with zero mean curvature in a hyperquadric Q.*

Proof. We assume that Z is either orthogonal or tangent to M. In the former case M is totally umbilical by Corollary 4.6. Recall that in virtue of Corollary 2.6 the vector field Z is of the form $Z(x) = U + \varphi x$. It is well known that the hyperquadrics and the hyperplanes are the only totally umbilical hypersurfaces in $\mathbb{R}^{n+2}_s$ that are orthogonal to such vector fields Z.

Moreover, the hyperquadrics are orthogonal to radial vector fields which are closed and conformal, whereas every hyperplane is orthogonal to a parallel vector field [3]. (Notice that totally umbilical submanifolds of the form (4) in Proposition 3.6 in [3] are not orthogonal to a parallel vector field.) In the later case, namely, when Z is tangent to M, we have two possibilities, depending whether Z is parallel or radial.

1. When Z is parallel, M is a cylinder over a CMC hypersurface $N \subset \sigma$, where σ is some translation of the hyperplane $\sigma := (\text{span } Z)^\perp$. This cylinder M is isometric to $N \times \mathbb{R}$ where $\mathbb{R}$ has either a Riemannian or Lorentzian metric depending whether Z is spacelike or timelike, respectively. Since M is a product, Z is a principal direction of M with associated principal curvature equal to zero. This implies that N has constant mean curvature in σ.
2. When Z is radial, i.e. $Z(x) = U + ax$ for some $a \neq 0 \in \mathbb{R}$ and $U \in \mathbb{R}^{n+2}_s$. Since Z is tangent, M is an open part of a hypercone $C := a \cdot C_\epsilon\Sigma + U$ where Σ is a hypersurface in a hyperquadric either $Q = \mathbb{S}^{n+1}_s(1)$ or $Q = \mathbb{H}^{n+1}_{s-1}(1)$ of $\mathbb{R}^{n+2}_s$. Then by Lemma 4.8, we conclude that our hypercone C has zero mean curvature in $\mathbb{R}^{n+2}_s$ and Σ has zero mean curvature in the hyperquadric Q. □

5. CMC constant angle surfaces

In this section we study the case of CMC semi-Riemannian surfaces M isometrically immersed in $\overline{\mathbb{M}}^3_s(c)$ making constant angle with respect to a closed and conformal vector field Z. We will focus on the Simons' type formula (10) derived in Theorem 4.1 and show that under our hypothesis it gives rise to a polynomial expression in the variable $y = \langle Z, \xi\rangle$. We begin by finding adequate expressions for $\langle \alpha, \alpha\rangle$ and $\Delta\langle Z, \xi\rangle$ in Lemmas 5.1 and 5.3.

Lemma 5.1.

$$\langle \alpha, \alpha\rangle = \frac{2\epsilon_\xi \mu^4 \varphi^2}{\langle Z, \xi\rangle^2} + \epsilon_\xi \langle H, \xi\rangle^2 + \frac{2\epsilon_\xi \epsilon_Z \mu^2 \langle H, \xi\rangle \varphi}{\langle Z, \xi\rangle}.$$

Proof. Let $W \in \Gamma(TM)$ with $\langle W, W\rangle = \epsilon_W$ and $\langle W, T\rangle = 0$, where $T = Z^\top/|Z^\top|$. Then

$$\epsilon_W \alpha(W,W) + \epsilon_T \alpha(T,T) = H = \epsilon_\xi \langle H, \xi\rangle \xi,$$

and by Remark 3.6

$$\begin{aligned}\alpha(W,W) &= \epsilon_W \left(\epsilon_\xi \langle H,\xi\rangle \xi - \epsilon_T \alpha(T,T)\right)\\ &= \epsilon_W \left(\epsilon_\xi \langle H,\xi\rangle + \frac{\epsilon_\xi \epsilon_Z \varphi \langle Z,\xi\rangle}{|Z|^2}\right)\xi.\end{aligned}$$

Therefore,

$$\begin{aligned}\langle \alpha,\alpha\rangle &= \sum_{i,j} \epsilon_i\epsilon_j \langle \alpha(e_i,e_j), \alpha(e_i,e_j)\rangle\\ &= \langle \alpha(T,T),\alpha(T,T)\rangle + \langle \alpha(W,W),\alpha(W,W)\rangle\\ &= \frac{\varphi^2\langle Z,\xi\rangle^2}{|Z|4}\epsilon_\xi + \left(\epsilon_\xi\langle H,\xi\rangle + \frac{\epsilon_\xi\epsilon_Z\varphi\langle Z,\xi\rangle}{|Z|^2}\right)^2 \epsilon_\xi\\ &= \epsilon_\xi \left(\frac{\varphi^2\langle Z,\xi\rangle^2}{|Z|^4} + \langle H,\xi\rangle^2 + 2\epsilon_Z \frac{\varphi\langle H,\xi\rangle\langle Z,\xi\rangle}{|Z|^2} + \frac{\varphi^2\langle Z,\xi\rangle^2}{|Z|^4}\right)\\ &= \frac{2\epsilon_\xi\varphi^2\langle Z,\xi\rangle^2}{|Z|^4} + \epsilon_\xi\langle H,\xi\rangle^2 + \frac{2\epsilon_\xi\epsilon_Z\varphi\langle H,\xi\rangle\langle Z,\xi\rangle}{|Z|^2}\\ &= \frac{2\epsilon_\xi\mu^4\varphi^2}{\langle Z,\xi\rangle^2} + \epsilon_\xi\langle H,\xi\rangle^2 + \frac{2\epsilon_\xi\epsilon_Z\mu^2\langle H,\xi\rangle\varphi}{\langle Z,\xi\rangle}. \qquad \square\end{aligned}$$

Proposition 5.2. *Let us assume that $c = 0$, i.e. $\overline{\mathbb{M}}^3_s(c)$ is the semi-Euclidean space $\mathbb{R}^3_s$. If Z is parallel and it is neither orthogonal nor tangent to M, then M is a portion of a plane.*

Proof. Let us observe that in virtue of Corollary 2.6 we have $\varphi = 0$. We know by Corollary 4.2 that $\langle \alpha,\alpha\rangle = 0$. Therefore, by Lemma 5.1 we have $\epsilon_\xi\langle H,\xi\rangle^2 = 0$ and thus $H = 0$. Consider now and orthonormal frame $\{T, W\}$ in M like in Corollary 4.7. Hence

$$0 = H = \epsilon_W \alpha(W,W) + \epsilon_T \alpha(T,T).$$

And we know by Proposition 3.5 that $\alpha(T,T) = \alpha(T,W) = 0$, so $\alpha = 0$. Thus M is totally geodesic, hence a plane in $\mathbb{R}^3_1$. $\square$

Now, we are going to obtain the Laplacian of $\langle Z, \xi\rangle$.

Lemma 5.3.

$$\Delta\langle Z,\xi\rangle = -c\langle Z,\xi\rangle + c\epsilon_Z\epsilon_\xi\mu^2\langle Z,\xi\rangle + \epsilon_Z\epsilon_\xi\mu^2\varphi\langle H,\xi\rangle + \frac{(\epsilon_Z + \epsilon_\xi\mu^2)\mu^2\varphi^2}{\langle Z,\xi\rangle}.$$

Proof. Due to the fact that

$$\langle \nabla\varphi, Z\rangle = \langle -cZ^\top, Z\rangle = -c\langle Z^\top, Z^\top\rangle = -c\epsilon_T \frac{\lambda^2}{\mu^2}\langle Z,\xi\rangle^2,$$

we obtain

$$\begin{aligned}\Delta\langle Z,\xi\rangle &= \frac{\langle Z,\xi\rangle}{|Z|^2}\left(\epsilon_Z(\langle\nabla\varphi, Z\rangle + \varphi\langle H,Z\rangle + 2\varphi^2) - \frac{\varphi^2\langle Z^\top, Z^\top\rangle}{|Z|^2}\right)\\ &= \frac{\mu^2}{\langle Z,\xi\rangle}\left(-c\epsilon_Z\epsilon_T\frac{\lambda^2}{\mu^2}\langle Z,\xi\rangle^2 + \epsilon_Z\epsilon_\xi\varphi\langle H,\xi\rangle\langle Z,\xi\rangle + 2\epsilon_Z\varphi^2 - \epsilon_T\lambda^2\varphi^2\right)\\ &= -c\epsilon_Z\epsilon_T\lambda^2\langle Z,\xi\rangle + \epsilon_Z\epsilon_\xi\mu^2\varphi\langle H,\xi\rangle + \frac{(2\epsilon_Z - \epsilon_T\lambda^2)\mu^2\varphi^2}{\langle Z,\xi\rangle}\\ &= -c(1-\epsilon_Z\epsilon_\xi\mu^2)\langle Z,\xi\rangle + \epsilon_Z\epsilon_\xi\mu^2\varphi\langle H,\xi\rangle + \frac{(2\epsilon_Z + \epsilon_\xi\mu^2 - \epsilon_Z)\mu^2\varphi^2}{\langle Z,\xi\rangle}\\ &= -c\langle Z,\xi\rangle + c\epsilon_Z\epsilon_\xi\mu^2\langle Z,\xi\rangle + \epsilon_Z\epsilon_\xi\mu^2\varphi\langle H,\xi\rangle + \frac{(\epsilon_Z + \epsilon_\xi\mu^2)\mu^2\varphi^2}{\langle Z,\xi\rangle}. \qquad\square\end{aligned}$$

Lemma 5.4. *The function $y = \langle Z,\xi\rangle$ satisfies the relation*

$$\left(\epsilon_\xi a^2 - c\epsilon_Z(\epsilon_Z - \epsilon_\xi\mu^2)\right)y^2 + \epsilon_Z ab\varphi y + b\mu^2\varphi^2 = 0, \tag{13}$$

where $a = \langle H,\xi\rangle$ and $b = \epsilon_Z + 3\epsilon_\xi\mu^2$.

Proof. By the Simons' type formula (10), it follows that

$$\begin{aligned}0 = &-c(1-\epsilon_Z\epsilon_\xi\mu^2)\langle Z,\xi\rangle + \epsilon_Z\epsilon_\xi\mu^2\varphi\langle H,\xi\rangle + \frac{(\epsilon_Z + \epsilon_\xi\mu^2)\mu^2\varphi^2}{\langle Z,\xi\rangle}\\ &+ \frac{2\epsilon_\xi\mu^4\varphi^2}{\langle Z,\xi\rangle} + \epsilon_\xi\langle H,\xi\rangle^2\langle Z,\xi\rangle + 2\epsilon_\xi\epsilon_Z\mu^2\langle H,\xi\rangle\varphi + \varphi\langle H,\xi\rangle.\end{aligned}$$

Therefore

$$\begin{aligned}0 = &\left(\epsilon_\xi\langle H,\xi\rangle^2 - c(1-\epsilon_Z\epsilon_\xi\mu^2)\right)\langle Z,\xi\rangle + (3\epsilon_Z\epsilon_\xi\mu^2 + 1)\varphi\langle H,\xi\rangle\\ &+ \frac{(\epsilon_Z + 3\epsilon_\xi\mu^2)\mu^2\varphi^2}{\langle Z,\xi\rangle},\end{aligned}$$

which implies that

$$\begin{aligned}0 = &\left(\epsilon_\xi\langle H,\xi\rangle^2 - c\epsilon_Z(\epsilon_Z - \epsilon_\xi\mu^2)\right)\langle Z,\xi\rangle^2\\ &+ \epsilon_Z(3\epsilon_\xi\mu^2 + \epsilon_Z)\langle H,\xi\rangle\varphi\langle Z,\xi\rangle + (\epsilon_Z + 3\epsilon_\xi\mu^2)\mu^2\varphi^2.\end{aligned}$$

Now, by letting $a = \langle H,\xi\rangle$, $b = \epsilon_Z + 3\epsilon_\xi\mu^2$ and $y = \langle Z,\xi\rangle$, we arrive at the desired expression. $\square$

Theorem 5.5. *Let M be a semi-Riemannian CMC hypersurface in $\overline{\mathbb{M}}^3_s(c)$ making a constant angle with a closed and conformal vector field Z. If Z is neither tangent nor orthogonal to M then the function $y = \langle Z,\xi\rangle$ satisfies a polynomial equation of degree at most 4.*

Proof. We analyze the case $c = 0$ first. In this case, Equation (13) in Lemma 5.4 gives at once

$$\epsilon_\xi a^2y^2 + \epsilon_Z ab\varphi y + b\mu^2\varphi^2 = 0. \tag{14}$$

Notice that since M has constant angle, Lemma 3.3 guarantees that both μ^2 and b are constant. Moreover, since M is CMC then a is constant as well. Finally, since $c = 0$ then φ is constant by Corollary 2.6. Thus Equation (14) is indeed a polynomial in y of degree at most 2.

Now we deal with the case $c \neq 0$. Let us consider the equivalent form of Equation (13)

$$\left(\epsilon_\xi a^2 - c\epsilon_Z(\epsilon_Z - \epsilon_\xi \mu^2)\right) y^2 + b\mu^2\varphi^2 = -\epsilon_Z ab\varphi y,$$

and square both sides to obtain

$$\begin{aligned} a^2b^2\varphi^2y^2 = &\left(\epsilon_\xi a^2 - c\epsilon_Z(\epsilon_Z - \epsilon_\xi\mu^2)\right)^2 y^4 \\ &+ 2\left(\epsilon_\xi a^2 - c\epsilon_Z(\epsilon_Z - \epsilon_\xi\mu^2)\right) b\mu^2\varphi^2y^2 + b^2\mu^4\varphi^4. \end{aligned}$$

Thus we have

$$\delta^2 y^4 + (2\delta b\mu^2 - a^2b^2)\varphi^2y^2 + b^2\mu^4\varphi^4 = 0, \tag{15}$$

where

$$\delta = \epsilon_\xi a^2 - c\epsilon_Z(\epsilon_Z - \epsilon_\xi\mu^2)$$

is a constant. Now, using Corollary 2.6 we can express φ^2 in terms of $y = \langle Z, \xi\rangle$ as follows: First note that $d = \langle U, U\rangle = \langle Z, Z\rangle + \varphi^2/2$ is constant since $U = Z - \varphi x$ is parallel. Thus

$$\varphi^2 = c\left(d - \frac{\epsilon_Z}{\mu^2}\langle Z, \xi\rangle^2\right) = c\left(d - \frac{\epsilon_Z}{\mu^2}y^2\right)$$

and

$$\varphi^4 = c^2\left(d^2 - \frac{2d\epsilon_Z}{\mu^2}y^2 + \frac{1}{\mu^4}y^2\right).$$

Then from Equation (15) we obtain

$$\delta^2y^4 + (2\delta b\mu^2 - a^2b^2)c\left(d - \frac{\epsilon_Z}{\mu^2}y^2\right)y^2 + b^2\mu^4c^2\left(d^2 - \frac{2d\epsilon_Z}{\mu^2}y^2 + \frac{y^4}{\mu^4}\right) = 0$$

which simplifies to

$$\beta y^4 + \gamma y^2 + b^2\mu^4c^2d^2 = 0, \tag{16}$$

where

$$\beta = \delta^2 - c(2\delta b\mu^2 - a^2b^2)\frac{\epsilon_Z}{\mu^2} + b^2c^2,$$

and

$$\gamma = cd(2\delta b\mu^2 - a^2b^2) - 2\epsilon_Z db^2\mu^2c^2$$

are constant. Thus (16) is a polynomial of degree at most 4 and the proof is complete. □

Now we are ready to extract some conclusions from Theorem 5.5. As we have done before, we deal with the $c = 0$ case first.

Corollary 5.6. *Let M be a CMC semi-Riemannian surface isometrically immersed in $\mathbb{R}^3_s$ and making a constant angle with respect to a closed and conformal vector field Z. Then either*

1. *M is an open portion of a plane.*
2. *M is an open portion of a quadric.*
3. *M is an open portion of a cylinder in direction Z over a curve with constant curvature in a plane orthogonal to Z.*

This means that M is isoparametric.

Proof. If Z is orthogonal to M then Corollary 4.6 establishes that M is totally umbilical, hence, it is an open portion of a quadric when Z is radial, or a plane when Z is parallel. Further, if Z is tangent to M then by Corollary 4.9, we have two possibilities:

1. If Z is parallel, then M is a cylinder over a CMC curve in a plane orthogonal to Z. The generator of the cylinder is parallel to Z and the curve has constant curvature in the plane – so the curve is a circle, a hyperbola or a line.
2. If Z is radial then M is a cone over a minimal curve in a quadric. This means that the curve is a geodesic. So, M is a radial cone over a geodesic in a quadric and therefore M is a plane. Let us recall that the geodesics in a quadric are the intersection of a quadric with a plane across the center of the quadric.

Now, let us assume that Z is neither orthogonal nor tangent to M, then Equation (14) holds. If polynomial (14) does not vanish identically, it follows that $y = \langle Z, \xi \rangle$ is a constant, being the root of a polynomial. Since M has constant angle, we have by definition that $\langle Z/|Z|, \xi \rangle$ is constant and therefore $|Z|$ is a constant as well. By Lemma 3.4, we have that $\varphi \equiv 0$ since Z is not orthogonal to M by hypothesis. Thus we have that Z is parallel and Proposition 5.2 asserts that M is an open portion of a plane.

The only case remaining occurs when polynomial (14) vanishes identically. By looking at the coefficients we immediately see that this happens if and only if $a = \langle H, \xi \rangle = 0$ and either $b = \epsilon_Z + 3\epsilon_\xi \mu^2 = 0$ or $\varphi \equiv 0$. The latter case has been analyzed already, whereas the former case yields $H = 0$ and $\mu^2 = -\epsilon_Z \epsilon_\xi / 3$. This implies that $\epsilon_Z = -\epsilon_\xi$ and thus the ambient space can not be Euclidean, so it should be the Minkowski space $\mathbb{R}^3_1$ or else $\mathbb{R}^3_2$. In either case, relation (8) in Lemma 3.3 guarantees that $\epsilon_T = \epsilon_Z$. Therefore we have two cases: either M is timelike and then $\epsilon_\xi = 1$, $\epsilon_T = -1 = \epsilon_Z$; or else M is spacelike and then $\epsilon_\xi = -1$, $\epsilon_T = 1 = \epsilon_Z$. In the Minkowski case both options imply that M is a portion of a plane (refer to Remark 3.6 in p. 1107 of [15] for the timelike case and Remark 3.6 in p. 218 of [14] for the spacelike case). For the $\mathbb{R}^3_2$ case we can reverse causality in order to obtain the same results. □

For the case $c \neq 0$ we have a similar result.

Corollary 5.7. *Let $c \neq 0$ and M be a CMC semi-Riemannian surface isometrically immersed in $\overline{\mathbb{M}}^3_s(c)$ and making a constant angle with respect to a closed and conformal vector field Z. Then either*

1. *M is totally umbilic.*
2. *M is totally geodesic.*
3. *M satisfies the following conditions*

$$c\epsilon_\xi > 0, \quad \epsilon_Z = \epsilon_T = -\epsilon_\xi, \quad \mu^2 = \frac{1}{3}, \quad \lambda^2 = \frac{4}{3} \quad \text{and} \quad \langle H, \xi \rangle^2 = \frac{4}{3} c\epsilon_\xi.$$

Proof. As before, let us assume first that Z is neither orthogonal nor tangent to M. Thus, if (16) does not vanish identically, then $y = \langle Z, \xi \rangle$ is a constant, being one of its roots. As a consequence, $|Z|$ is constant in M. Then Lemma 2.7 will give rise to a contradiction: if $\varphi \equiv 0$ then $c = 0$ by Lemma 2.5; otherwise,

Z is orthogonal to M, contradicting our assumption. Alternatively, if Z is orthogonal to M we have that M is totally umbilical by Corollary 4.6, whereas, if Z is tangent to M we assert M is totally geodesic by Corollary 4.7. Finally, let us observe that the polynomial (16) vanishes precisely when $\delta = 0$ and $b = 0$ (notice that the constants c and d do not vanish and we can assume $\mu \neq 0$ as well). These latter conditions are equivalent to

$$\mu^2 = -\frac{\epsilon_Z \epsilon_\xi}{3} \quad \text{and} \quad \langle H, \xi \rangle^2 = \frac{4}{3} c \epsilon_\xi,$$

and hence we must have the inequality $c\epsilon_\xi > 0$. The result follows from the relation (8) in Lemma 3.3. □

Remark 5.8. The above Corollary 5.7 proves that if the mean curvature of the surface is not $\pm 2/\sqrt{3}$ they are either totally umbilic or totally geodesic. In particular, when the surface has zero mean curvature it is totally geodesic.

The following result is an immediate consequence of Corollary 5.7:

Corollary 5.9. *Let M be a CMC semi-Riemannian surface isometrically immersed in a three dimensional space form of non-vanishing curvature and making a constant angle with respect to a closed and conformal vector field Z. Then M is totally umbilical or totally geodesic provided any of the following conditions hold:*

1. *$M \subset \mathbb{S}^3_1(r)$ is spacelike.*
2. *$M \subset \mathbb{H}^3_1(r)$ is timelike.*
3. *Z and ξ have the same causality.*

Proof. If M satisfies (1) or (2) then we have $c\epsilon_\xi < 0$, whereas if M satisfies (3) then $\epsilon_Z = \epsilon_\xi$ holds. □

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References

[1] C. Barrera Cadena, A.J. Di Scala, G. Ruiz-Hernández, Helix surfaces in Euclidean spaces, Beitr. Algebra Geom. 56 (2015) 551–573.
[2] A. Barros, A. Brasil, A. Caminha, Stability of spacelike hypersurfaces in foliated space-times, Differ. Geom. Appl. 26 (2008) 357–365.
[3] B.Y. Chen, Pseudo Riemannian Geometry δ-Invariants and Applications, World Publishing Scientific, 2011.
[4] D. Chen, G. Chen, H. Chen, F. Dillen, Constant angle surfaces in $\mathbb{S}^3 \times \mathbb{R}$, Bull. Belg. Math. Soc. Simon Stevin 19 (2012) 289–304.
[5] F. Dillen, J. Fastenakels, J. Van der Veken, Surfaces in $\mathbb{S}^2 \times \mathbb{R}$ with a canonical principal direction, Ann. Glob. Anal. Geom. 35 (2009) 381–396.
[6] F. Dillen, J. Fastenakels, J. Van der Veken, L. Vrancken, Constant angle surfaces in $\mathbb{S}^2 \times \mathbb{R}$, Monatshefte Math. 152 (2007) 89–96.
[7] F. Dillen, M.I. Munteanu, Constant angle surfaces in $\mathbb{H}^2 \times \mathbb{R}$, Bull. Braz. Math. Soc. (N. S.) 40 (2009) 85–97.
[8] F. Dillen, M.I. Munteanu, A.I. Nistor, Canonical coordinates and principal directions for surfaces in $\mathbb{H}^2 \times \mathbb{R}$, Taiwan. J. Math. 15 (2011) 2265–2289.
[9] F. Dillen, M.I. Munteanu, J. Van der Veken, L. Vrancken, Classification of constant angle surfaces in a warped product, Balk. J. Geom. Appl. 16 (2011) 35–47.

[10] D. Eardley, J. Isenberg, J. Marsden, V. Moncrief, Homothetic and conformal symmetries of solutions to Einstein's equations, Commun. Math. Phys. 106 (1986) 137–158.
[11] D. Fetcu, A classification result for helix surfaces with parallel mean curvature in product spaces, Ark. Mat. 53 (2) (2015) 249–258.
[12] D. Fetcu, H. Rosenberg, Surfaces with parallel mean curvature in $\mathbb{S}^3 \times \mathbb{R}$ and $\mathbb{H}^3 \times \mathbb{R}$, Mich. Math. J. 61 (2012) 715–729.
[13] Y. Fu, A.I. Nistor, Constant angle property and canonical principal directions for surfaces in $\mathbb{M}^2(c) \times \mathbb{R}_1$, Mediterr. J. Math. 10 (2013) 1035–1049.
[14] Y. Fu, D. Yang, On constant slope spacelike surfaces in 3-dimensional Minkowski space, J. Math. Anal. Appl. 385 (2012) 208–220.
[15] Y. Fu, X. Wang, Classification of timelike constant slope surfaces in 3-dimensional Minkowski space, Results Math. 63 (2013) 1095–1108.
[16] E. Garnica, O. Palmas, G. Ruiz-Hernández, Classification of constant angle hypersurfaces in warped products via eikonal functions, Bol. Soc. Mat. Mexicana 18 (2012) 29–41.
[17] F. Güler, G. Saffak, E. Kasap, Timelike constant angle surfaces in Minkowski space $\mathbb{R}^3_1$, Int. J. Contemp. Math. Sci. 6 (2011) 2189–2200.
[18] P. Jordan, J. Ehlers, W. Kundt, Republication of: Exact solutions of the field equations of the general theory of relativity, Gen. Relativ. Gravit. 41 (2009) 2191–2280.
[19] W. Kühnel, H. Rademacher, Conformal vector fields on pseudo-Riemannian spaces, Differ. Geom. Appl. 7 (1997) 237–250.
[20] R. Lopez, M.I. Munteanu, Constant angle surfaces in Minkowski space, Bull. Belg. Math. Soc. Simon Stevin 18 (2011) 271–286.
[21] S. Montiel, Uniqueness of spacelike hypersurfaces of constant mean curvature in foliated spacetimes, Math. Ann. 314 (1999) 529–553.
[22] M.I. Munteanu, A.I. Nistor, Surfaces in $\mathbb{E}^3$ making constant angle with Killing vector fields, Int. J. Math. 23 (2012) 1250023.
[23] B. O'Neill, Semi-Riemannian Geometry, Academic Press, Inc., 1983.
[24] O. Palmas, G. Ruiz-Hernández, Spacelike hypersurfaces with a canonical principal direction, in: J. Van der Veken, I. Van de Woestyne, L. Verstraelen, L. Vrancken (Eds.), Pure and Applied Differential Geometry PADGE 2012, Shaker Verlag, 2012.
[25] T.K. Pan, Conformal vector fields in compact Riemannian manifolds, Proc. Am. Math. Soc. 14 (1963) 653–657.
[26] J. Simons, Minimal varieties in Riemannian manifolds, Ann. Math. 88 (1968) 62–105.
[27] Y. Xin, Geometry of Harmonic Maps, Progress in Nonlinear Differential Equations and Their Applications, vol. 23, Birkhäuser, 1996.