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On the geometry of null hypersurfaces in Minkowski space

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ABSTRACT

The present work is divided into three parts. First we study the null hypersurfaces of the Minkowski space $\mathbb{R}^{n+2}_1$, classifying all rotation null hypersurfaces in $\mathbb{R}^{n+2}_1$. In the second part we start our analysis of the submanifold geometry of the null hypersurfaces. In the particular case of the $(n+1)$-dimensional light cone, we characterize its totally umbilical spacelike hypersurfaces, show the existence of non-totally umbilical ones and give a uniqueness result for the minimal spacelike rotation surfaces in the 3-dimensional light cone. In the third and final part we consider an isolated umbilical point on a spacelike surface immersed in the 3-dimensional light cone of $\mathbb{R}^4_1$ and obtain the differential equation of the principal configuration associated to this point, showing that every classical generic Darbouxian principal configuration appears in this context.

1. Introduction

One of the main features that distinguishes Lorentzian geometry from its Riemannian counterpart is the existence of null submanifolds. Let us recall that an immersed submanifold of a Lorentzian manifold in general will not inherit a metric structure from its ambient space since its first fundamental form may not have a constant signature, and consequently there may be points in which it degenerates.

Let (M, g) be a Lorentzian manifold and $i : N \to M$ be an immersion. We say that (N, h) is a *null submanifold* of (M, g) if the tensor $h = i^*g$ is degenerate at every point; that is, if $h_p : T_pM \times T_pM \to \mathbb{R}$ is a degenerate symmetric bilinear form.

Null submanifolds are of great interest in the context of General Relativity, since some of the most relevant concepts in the physical theory find mathematical realizations as null submanifolds of space–time. For instance, we have that free falling particles are represented by null geodesics and event horizons by null hypersurfaces. Other objects often related to the causal structure of space–time, though not smooth in general, have smooth null portions. Such is the case of achronal boundaries and Cauchy horizons. Another example comes from considering the end of an asymptotically flat space–time, which is conceived as a null manifold boundary (hence a null hypersurface) of space–time; see [1,2].

In spite of the importance of null geometry in General Relativity, a systematic and formal study of the geometry of null submanifolds was missing until the middle of the 1980's (see [3] and Chapter 2 of [4] for a nice introduction to the subject). Since then, a renewed interest to study null submanifolds for its own sake has sparkled in the mathematical community. In

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particular, the study of spacelike surfaces immersed in three dimensional null submanifolds has drawn a lot of attention in recent years. As it turns out, this topic has a strong connection to a physical scenario: it models the surface of a black hole within its event horizon [1,2]. Since classical solutions to Einstein's field equations (like the Kerr–Schwarzschild family) have totally geodesic event horizons, this scenario is somewhat parallel, from a geometric point of view, to the classical theory of surfaces immersed in the three dimensional Euclidean space, the main difference being that the normal direction to the immersed spacelike surface is null. Spacelike surfaces immersed in null hypersurfaces are also related to the singularity theory in General Relativity, since they represent various kinds of trapped surfaces, that is, surfaces that lie in a region of space–time where the gravitational field is strong enough to cause the focusing of future null geodesics emanating from them, a phenomenon that points to the development of gravitational collapse and the occurrence of singularities [1,2,5].

To this date, there is a fair amount of research devoted to the study of the geometry of submanifolds of semi-Riemannian manifolds in a spirit closely related to their Riemannian analogues when the induced metric on the submanifold is nondegenerate. The case in which the induced metric is degenerate (null) had received some attention also. For example, in [6,7] Kupeli develops the intrinsic geometry of null submanifolds, paying special attention to null curves and hypersurfaces. This approach turned out very useful in establishing a local comparison theory for null hypersurfaces, via Galloway's Maximum Principle for null hypersurfaces [8]. As consequences of this latter result we have the Null Splitting Theorem [8] and various space–time uniqueness theorems [9,10].

Complementing Kupeli's and Galloway's intrinsic approach, Bejancu and Duggal in [11,12] developed an extrinsic method to study the null hypersurfaces immersed in semi-Euclidean spaces based on the construction of a null transversal vector bundle. In the case of Monge null hypersurfaces, the existence of canonical screen distributions enables the study of a large class of codimension two spacelike surfaces immersed in null hypersurfaces [13]. This approach has proved useful in many contexts, for instance, in characterizing the degenerate foliations in three dimensional ambient spaces [14] or totally umbilical null hypersurfaces in manifolds of constant curvature [15]. For an up-to-date account of results, see [16]. The basic framework of our present work relies heavily on this extrinsic approach.

The study of Gauss maps for null hypersurfaces and codimension two spacelike submanifolds of Minkowski space dates back to the work of Kossowski [17,18] and was greatly enhanced by the introduction of techniques stemming from the singularity theory and contact geometry due to Izumiya, Romero-Fuster and their collaborators; their work paved the way to study the geometry of flat null hypersurfaces and its invariants; see [19–21]. Important results pertaining totally umbilical and totally semi-umbilical codimension two spacelike surfaces have also been achieved using this approach [22,23]. More recently, Izumiya applied novel techniques as Legendrian dual fibrations in the study of codimension two spacelike surfaces immersed in the light cone [24–26].

Recall that for a hypersurface S immersed in a Euclidean space with normal vector field N the shape operator A_N is self-adjoint and thus there are well defined line fields corresponding to the directions defined by the eigenvectors of A_N. The integral curves of these line fields form the so-called *N-principal curvature lines*, which have singularities precisely on the umbilical points of S. The study of the dynamics of the principal configurations around isolated umbilical points is an active field of research, in which seminal contributions have been made in [27,28,30] for Euclidean ambient spaces. See [31] for a historical account in the Euclidean setting from the work of Monge (1796) up to recent years. On the other hand, there are only a few works dealing with principal configurations in the Lorentzian setting [32,33].

In this work we will be mainly concerned with the study of spacelike submanifolds immersed in null hypersurfaces of the $(n+2)$-dimensional Minkowski space. The paper is organized as follows: in Section 2 we establish the basic set up and definitions. In Section 3 we define and classify the null rotation hypersurfaces of the Minkowski space (see Theorem 3.1):

Let M be a connected null rotation hypersurface in $\mathbb{R}^{n+2}_1$. Then M is an open subset of either:

- the light cone Λ^{n+1}_0;
- a cylinder over a n-dimensional light cone Λ^n_0; or
- a null hyperplane.

Henceforth we consider a null hypersurface of the Minkowski space as our environment and begin the study of its submanifold geometry. In Section 4 we focus on the $(n+1)$-dimensional light cone and characterize its totally umbilical hypersurfaces (see Proposition 4.1):

Let $S \subset \Lambda^{n+1}_0$ be a spacelike hypersurface of the light cone. S is U-totally umbilical with respect to any normal vector field U if and only if S is the intersection of the cone with a $(n+1)$-dimensional hyperplane not passing through the origin.

We also give a criterion for a rotation surface $S \subset \Lambda^3_0 \subset \mathbb{R}^4_1$ to be totally umbilical (see Proposition 4.3):

Let $\Phi(u,v) = (x_0(u), x_1(u), x_2(u)\cos v, x_2(u)\sin v)$ be the parametrization of a spacelike spherical rotation surface $S \subset \Lambda^3_0 \subset \mathbb{R}^4_1$. Then S is totally umbilical if and only if

$$\frac{x_1'}{x_2} = -x_1'x_2'' + x_1''x_2'.$$

This result may be seen also as a method to give examples of non-totally umbilical spacelike hypersurfaces of the light cone. In [34], Liu gave an example of a surface of this type (see Eq. (18)) and proved it is the only surface in Λ^3_0 being homogeneous and non-totally umbilical. We prove further that Liu's example is the only spacelike minimal rotation surface in the three-dimensional cone (see Theorem 4.4):

Let $S \subset \Lambda_0^3$ be a spacelike minimal rotation surface. Then *S can be parametrized locally as* $\Phi(u, v) = (\cosh u, \sinh u, \cos v, \sin v)$.

Finally, in Section 5 we obtain the differential equation of the principal curvature lines in a neighborhood of an isolated umbilical point for a generic spacelike surface immersed in the 3-dimensional light cone of the 4-dimensional Minkowski space (see Proposition 5.1):

The 1*-jet of the differential equation of* η*-principal curvature lines for a generic spacelike surface S immersed in the light cone* Λ_0^3 of $\mathbb{R}_1^4$ *is given by*

$$A_1(x, y)\, dy^2 + B_1(x, y)\, dx\, dy - A_1(x, y)\, dx^2 = 0,$$

where

$$A_1(x, y) = \frac{dx + by}{f_o} + \frac{2\alpha f_{xo} y}{f_o^2}$$

and

$$B_1(x, y) = \frac{(a - b)x + (d - c)y}{f_o} + \frac{4\alpha f_{xo} x}{f_o^2}.$$

The types of principal configurations which appear generically are therefore Darbouxian.

2. Preliminaries

The *Minkowski* $(n+2)$*-dimensional space* $\mathbb{R}_1^{n+2}$ is the $(n+2)$-dimensional vector space endowed with the scalar product

$$\langle p, q\rangle = -u_0 v_0 + \sum_{i=1}^{n+1} u_i v_i,$$

where $(u_0, u_1, \ldots, u_{n+1})$ and $(v_0, v_1, \ldots, v_{n+1})$ are respectively the coordinates of p and q relative to a basis $e_i, i = 0, 1, \ldots, n+1$.

Throughout this work M will denote a *null* (or *lightlike*) hypersurface of $\mathbb{R}_1^{n+2}$, that is, a hypersurface such that the restriction of the metric $\langle\ ,\ \rangle$ to the tangent bundle TM is degenerate. This degeneracy condition is equivalent to the existence of a vector field $\xi \in \Gamma(TM)$ everywhere different from zero such that $\langle \xi, X\rangle = 0$ for each $X \in \Gamma(TM)$.

As an important example of a null hypersurface, we define the *light cone* Λ_0^{n+1} of $\mathbb{R}_1^{n+2}$ by

$$\Lambda_0^{n+1} = \{p \in \mathbb{R}_1^{n+2} \mid \langle p, p\rangle = 0, p \neq 0\}.$$

In order to show that Λ_0^{n+1} is null, let $u_0, u_1, \ldots, u_{n+1}$ be the standard coordinates in $\mathbb{R}_1^{n+2}$ and $\partial_0, \partial_1, \ldots, \partial_{n+1}$ be the corresponding tangent vector fields. Then it is easy to see that the position vector field defined as

$$\xi = \xi(u_0, \ldots, u_{n+1}) = \sum_{i=0}^{n+1} u_i \partial_i$$

satisfies $\langle \xi, X\rangle = 0$ for each $X \in \Gamma(T\Lambda_0^{n+1})$. In particular, ξ is tangent and normal to Λ_0^{n+1}.

Given a null hypersurface $M \subset \mathbb{R}_1^{n+2}$, we will consider a *spacelike* hypersurface $S \subset M$, that is, $\dim S = n$ and the restriction of the metric of $\mathbb{R}_1^{n+2}$ to the tangent bundle TS is positive definite.

In order to study the geometry of S we first split the tangent bundle $T\mathbb{R}_1^{n+2}$ into three vector bundles. From [35] (see also [16]), we know that for each point in S there exists a neighborhood $\mathcal{U}$ in S and a vector field η defined in $\mathcal{U}$ such that

$$\langle \xi, \eta\rangle = 1, \qquad \langle \eta, \eta\rangle = \langle \eta, X\rangle = 0$$

for each $X \in \Gamma(TS|_{\mathcal{U}})$. We use this vector field η to write $T_p\mathbb{R}_1^{n+2}$ as

$$T_p\mathbb{R}_1^{n+2} = T_pM \oplus \operatorname{span}(\eta_p) \tag{1}$$

for each point $p \in \mathcal{U}$. Additionally, we decompose TM as

$$T_pM = T_pS \oplus_{\text{orth}} \operatorname{span}(\xi_p) \tag{2}$$

so that

$$T_p\mathbb{R}^{n+2}_1 = T_pS \oplus_{\mathrm{orth}} (\mathrm{span}(\xi_p) \oplus \mathrm{span}(\eta_p)).$$

Now we obtain the Gauss–Weingarten formulas. Denote by $\widetilde{\nabla}$ the semi-Riemannian connection in $\mathbb{R}^{n+2}_1$, and let $X, Y \in \Gamma(TM)$. Using the decomposition (1) we write the *first Gauss formula* as

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{3}$$

where ∇ denotes the induced connection in M; h will be called the *second fundamental form* of M in $\mathbb{R}^{n+2}_1$.

On the other hand, if $X \in \Gamma(TM)$, we use again (1) to write the *first Weingarten formula*

$$\widetilde{\nabla}_X \eta = -A_\eta X + \nabla^t_X \eta, \tag{4}$$

where A_η is the *shape operator* and ∇^t is the induced transversal connection of M in $\mathbb{R}^{n+2}_1$.

Let $P : TM \to TS$ be the orthogonal projection relative to the decomposition (2). Following [16], we establish the *second Gauss–Weingarten formulas* as

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY) \tag{5}$$

and

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{6}$$

for $X, Y \in \Gamma(TM)$, where ∇^* and ∇^{*t} are linear connections which will not be used here. We are interested only in the operator A^*_ξ and the form h^*, which are called the *screen shape operator* and the *screen second fundamental form*, respectively. It is easy to see that

$$\langle h^*(X, PY), \eta\rangle = \langle A_\eta X, PY\rangle; \tag{7}$$

compare for example, Eqs. (2.1.21) and (2.1.26) in [16].

Before proceeding with our study, we make some remarks.

Remark 2.1. Actually, Bejancu's and Duggal's approach is more general than ours: they choose a n-dimensional distribution in TM, called a *screen distribution* and establish the Gauss–Weingarten formulas in this setting; in particular, their screen distribution may not be integrable. In our case, since we are interested in *submanifolds* $S \subset M$, we take TS as the screen distribution, which is obviously integrable.

Remark 2.2. We may study the geometry of S directly as a submanifold of $\mathbb{R}^{n+2}_1$; in particular, we may consider a vector field U which is everywhere normal to S and define the U-shape operator A_U using the standard Weingarten formula. Since the null vector fields ξ and η are everywhere linearly independent, we may express any shape operator A_U in terms of the restrictions of A^*_ξ and A_η to TS. Explicitly, if $U = \lambda\xi + \mu\eta$ and $X \in \Gamma(TS)$, then

$$\begin{aligned}\widetilde{\nabla}_X U &= \widetilde{\nabla}_X(\lambda\xi + \mu\eta)\\ &= X(\lambda)\xi + \lambda\widetilde{\nabla}_X\xi + X(\mu)\eta + \mu\widetilde{\nabla}_X\eta\\ &= X(\lambda)\xi + \lambda(-A^*_\xi X + \nabla^\perp_X\xi) + X(\mu)\eta + \mu(-A_\eta X + \nabla^\perp_X\eta).\end{aligned}$$

By taking the part of the above expression which is tangent to S, we have

$$A_U X = \lambda A^*_\xi X + \mu A_\eta X. \tag{8}$$

Definition 2.3. The pair (M, S) is *screen conformal* whenever the shape operators A_η and A^*_ξ are linearly dependent at every point of S.

Example 2.4 (*See [16, p. 52]*). The upper half of the light cone Λ^{n+1}_0 may be written as the graph of the real function $F : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ given by

$$F(u_1, \dots, u_{n+1}) = \left(\sum_{i=1}^{n+1} u_i^2\right)^{1/2}. \tag{9}$$

Note that the position vector field $\xi \in \mathbb{R}^{n+2}_1$ satisfies $\widetilde{\nabla}_X\xi = X$ for any $X \in \Gamma(T\mathbb{R}^{n+2}_1)$. In particular, for *any* hypersurface $S \subset \Lambda^{n+1}_0$ and any $X \in \Gamma(T\Lambda^{n+1}_0)$ we have

$$A^*_\xi X = -PX. \tag{10}$$

In order to find a screen conformal pair (M, S), it is natural to consider the level hypersurfaces $F^{-1}(c)$, so let us fix one of them and call it S. At each $p \in S$, the tangent space T_pS is spanned by the vectors of the form

$$X = \sum_{i=1}^{n+1} a_i \partial_i, \quad \text{where } \sum_{i=1}^{n+1} a_i u_i = 0;$$

the second condition expresses the fact that $\langle X, \xi \rangle = 0$. The vector field

$$\eta = \frac{1}{2u_0^2} \left(-u_0 \partial_0 + \sum_{i=1}^{n+1} u_i \partial_i \right)$$

is null, everywhere normal to S and satisfies $\langle \eta, \xi \rangle = 1$. It is easy to see that the shape operators A^*_ξ and A_η satisfy

$$A_\eta X = \frac{1}{2u_0^2} A^*_\xi X$$

for each $X \in \Gamma(T\Lambda_0^{n+1})$. Hence, (Λ_0^{n+1}, S) is screen conformal.

To close this section, we define the concept of an umbilical point of S, which we analyze in detail in Sections 4 and 5. Roughly speaking, an umbilical point satisfies that its shape operator is a multiple of the identity operator.

Definition 2.5. A point $p \in S$ is η-*umbilical* if there exists a real-valued function k on S such that $A_\eta X(p) = k(p)X(p)$ for each $X \in \Gamma(TS)$. Analogously, p is ξ-*umbilical* if $A^*_\xi X(p) = \hat{k}(p)X(p)$ for each $X \in T_pS$.

If every point of S is η-umbilical (ξ-umbilical resp.), we say that S is η-totally umbilical (ξ-totally umbilical resp.).

Recalling our Example 2.4, Eq. (10) implies that any hypersurface S in the light cone is ξ-totally umbilical. On the other hand, we have seen that the pair (Λ_0^{n+1}, S) is screen conformal, where $S = F^{-1}(c)$ and F is given in (9), so that S is also η-totally umbilical. In fact, S is U-totally umbilical for *any* normal vector field U; for if $U = \lambda\xi + \mu\eta$ and $X \in \Gamma(TS)$, we have

$$A_U X = \left(\lambda + \frac{1}{2u_0^2}\mu \right) A^*_\xi X = -\left(\lambda + \frac{1}{2u_0^2}\mu \right) X.$$

The above argument applies in a more general setting: if (Λ_0^{n+1}, S) is screen conformal, then S is U-totally umbilical relative to each normal vector field U. We will prove later (see Proposition 4.1) a characterization of these U-totally umbilical hypersurfaces.

3. Null rotation hypersurfaces of $\mathbb{R}_1^{n+2}$

In order to get some insight into the realm of null hypersurfaces, we will give a brief description of the highly symmetric class of null rotation hypersurfaces. In fact, Inoguchi and Lee have proved in [36] that the only null rotation surfaces in $\mathbb{R}_1^3$ are (open subsets of) null hyperplanes or light cones. Here we analyze the problem in $\mathbb{R}_1^{n+2}$, showing that for $n > 1$ there is yet another class of null rotation hypersurfaces; namely, cylinders over light cones (see Theorem 3.1). For completeness we will give first the basic facts about rotation hypersurfaces.

A *rotation* of $\mathbb{R}_1^{n+2}$ is an isometry which leaves a line pointwise fixed. In [37], do Carmo and Dajczer classified the rotations of $\mathbb{R}_1^{n+2}$ into three classes, depending on the causal character of the fixed line: a rotation is said to be *spherical*, *hyperbolic* or *parabolic* if the fixed line is timelike, spacelike or null, respectively.

Consider a subgroup of isometries in $\mathbb{R}_1^{n+2}$ leaving a given line ℓ pointwise fixed, a two dimensional plane π containing ℓ and a curve γ in π which does not meet ℓ. Suppose further that the orbit of any point of γ under the subgroup has codimension 2. Then the orbit of γ is the *rotation hypersurface* of $\mathbb{R}_1^{n+2}$ *generated* by γ.

Now we are ready to characterize the null rotation hypersurfaces in $\mathbb{R}_1^{n+2}$. In the proof of our next result we summarize and use the explicit parametrizations of the rotation hypersurfaces given in [37].

We will denote by $e_0, \ldots, e_{n+1}$ the canonical orthonormal basis of $\mathbb{R}_1^{n+2}$, where $\langle e_0, e_0 \rangle = -1$.

Theorem 3.1. *Let M be a connected null rotation hypersurface in $\mathbb{R}_1^{n+2}$. Then M is an open subset of either:*

- *the light cone Λ_0^{n+1};*
- *a cylinder over a n-dimensional light cone Λ_0^n; or*
- *a null hyperplane.*

Proof. As mentioned before, our analysis depend on the causal character of the line fixed by the corresponding subgroup of rotations.

1. Consider first the case where the line fixed by the rotations is timelike. Specifically, we suppose that $\ell = \mathrm{span}\{e_0\}$ and that the curve γ is contained in $\mathrm{span}\{e_0, e_{n+1}\}$. As proved in [37] and up to isometries, a parametrization of the rotation hypersurface generated by γ may be written as

$$(t_1, \dots, t_n, s) \mapsto (x_0(s), x_{n+1}(s)\varphi_1, x_{n+1}(s)\varphi_2, \dots, x_{n+1}(s)\varphi_{n+1}), \tag{11}$$

where each φ_j is a function of $(t_1, \dots, t_n)$; $(\varphi_1, \dots, \varphi_{n+1})$ is an orthogonal parametrization of the unit sphere $\mathbb{S}^n$ and $\gamma(s) = (x_0(s), x_{n+1}(s))$. It is easy to see that Eq. (11) defines (locally) an immersion from an open subset of $\mathbb{R}^{n+1}$ into $\mathbb{R}_1^{n+2}$ if its differential has rank $n+1$, which in turn happens if and only if x_{n+1} does not vanish.

In order to obtain a null hypersurface, the determinant of the metric associated to the parametrization (11) must degenerate. We calculate the determinant of the metric as

$$\left(-(x_0'(s))^2 + (x_{n+1}'(s))^2\right) x_{n+1}^{2(n+1)}(s) \det(\mathbb{S}^n),$$

where $\det(\mathbb{S}^n)$ denotes the determinant of the standard metric on the unit sphere in terms of the parametrization $(\varphi_1, \dots, \varphi_{n+1})$, which is always different from zero. Hence, we obtain a null rotation hypersurface if and only if $x_{n+1}(s) \neq 0$ and

$$(x_0'(s))^2 = (x_{n+1}'(s))^2,$$

so that $x_0(s) = \pm x_{n+1}(s) + C$. Using a change of coordinates, we may suppose that $x_0(s) = \pm x_{n+1}(s)$, in which case the generating curve is a line with unit slope and the hypersurface is a light cone.

2. Now consider that the rotations fix a spacelike line; in fact, suppose that $\ell = \mathrm{span}\{e_{n+1}\}$. Following [38], we consider three cases, depending on the character of the plane π containing γ:

   (a) π is timelike: suppose that $\pi = \mathrm{span}\{e_0, e_{n+1}\}$, so that the generating curve has the form $\gamma(s) = (x_0(s), x_{n+1}(s))$. In analogy with the first case, define

   $$(t_1, \dots, t_n, s) \mapsto (x_0(s)\varphi_0, x_0(s)\varphi_1, \dots, x_0(s)\varphi_n, x_{n+1}(s)), \tag{12}$$

   where $(\varphi_0, \varphi_1, \dots, \varphi_n)$ is an orthogonal parametrization of the hyperbolic space $\mathbb{H}^n$ (represented as the upper branch of a hyperboloid in $\mathbb{R}_1^{n+1}$). In other words, the functions φ_j satisfy

   $$-\varphi_0^2 + \varphi_1^2 + \cdots + \varphi_n^2 = -1, \quad \varphi_0 > 0.$$

   Eq. (12) defines an immersion at every point where $x_0(s) \neq 0$. The metric of the corresponding hypersurface has determinant

   $$\left(-(x_0'(s))^2 + (x_{n+1}'(s))^2\right) x_0^{2(n+1)}(s) \det(\mathbb{H}^n);$$

   here $\det(\mathbb{H}^n)$ denotes the determinant of the metric of the hyperbolic space $\mathbb{H}^n$ in terms of the parametrization $(\varphi_0, \dots, \varphi_n)$. We obtain a null hypersurface if and only if $x_0(s) \neq 0$ and $x_0(s) = \pm x_{n+1}(s)$. The null rotation hypersurface is again a subset of the light cone, since the coordinates of the hypersurface satisfy

   $$-x_0^2\varphi_0^2 + x_0^2\varphi_1^2 + \cdots + x_0^2\varphi_n^2 + x_{n+1}^2 = -x_0^2 + x_{n+1}^2 = 0.$$

   (b) π is spacelike: Suppose $\pi = \mathrm{span}\{e_i, e_{n+1}\}$, where $i \neq 0, n+1$, so that $\gamma(s) = (x_i(s), x_{n+1}(s))$. The parametrization of the associated rotation hypersurface is

   $$(t_1, \dots, t_n, s) \mapsto (x_i(s)\varphi_1, x_i(s)\varphi_2, \dots, x_i(s)\varphi_{n+1}, x_{n+1}(s)), \tag{13}$$

   where $(\varphi_1, \dots, \varphi_{n+1})$ is an orthogonal parametrization of the hyperboloid of one sheet H^n satisfying $-\varphi_1^2 + \varphi_2^2 + \cdots + \varphi_{n+1}^2 = 1$, and $x_i(s)$ does not vanish. The determinant of the metric is

   $$\left((x_i'(s))^2 + (x_{n+1}'(s))^2\right) x_i^{2(n+1)}(s) \det(H^n).$$

   Since the above vanishes identically only when the generating curve degenerates to a point, in this case we can not obtain a null hypersurface.

   (c) π is null; that is, the restriction to π of the metric of $\mathbb{R}_1^{n+2}$ is degenerate. Here we may suppose that

   $$\pi = \mathrm{span}\left\{\frac{1}{\sqrt{2}}(e_i + e_0), e_{n+1}\right\},$$

   where $i \neq 0, n+1$. We write the expression of γ in this plane as $\gamma(s) = (x_i(s), x_{n+1}(s))$ and the orbit of γ is given by

   $$(t_1, \dots, t_n, s) \mapsto (x_i(s)\varphi_0, x_i(s)\varphi_1, \dots, x_i(s)\varphi_n, x_{n+1}(s)). \tag{14}$$

   In this case we take $(\varphi_0, \dots, \varphi_n)$ as an orthogonal parametrization of the upper half of the cone $\Lambda_0^n \subset \mathbb{R}_1^{n+1}$; that is,

   $$-\varphi_0^2 + \varphi_1^2 + \cdots + \varphi_n^2 = 0, \quad \varphi_0 > 0.$$

   Now $x_i(s) \neq 0$ in order to have an immersion. The determinant of the metric is

   $$(x_{n+1}')^2\, x_i^{2(n+1)} \det(\Lambda_0^n),$$

   where $\det(\Lambda_0^n)$ is the determinant of the standard metric on the light cone $\Lambda_0^n \subset \mathbb{R}_1^{n+1}$. Since the light cone is already a null hypersurface, this determinant vanishes everywhere, independently from the behavior of x_i and x_{n+1}. This hypersurface may be thought as the cylinder along the x_{n+1} axis over Λ_0^n.

3. If the rotations fix a null line, say, $\ell = \mathrm{span}\{e_{n+1} + e_0\}$, it will be convenient to use a *null frame* $\bar{e}_0, \dots, \bar{e}_{n+1}$ for $\mathbb{R}_1^{n+2}$, defined by

$$\bar{e}_0 = \frac{1}{\sqrt{2}}(e_{n+1} + e_0), \qquad \bar{e}_{n+1} = \frac{1}{\sqrt{2}}(e_{n+1} - e_0),$$

and $\bar{e}_i = e_i$ for $i = 1, \ldots, n$. This frame satisfies

$$\langle \bar{e}_0, \bar{e}_0 \rangle = \langle \bar{e}_{n+1}, \bar{e}_{n+1} \rangle = 0, \qquad \langle \bar{e}_0, \bar{e}_{n+1} \rangle = 1,$$

and $\langle \bar{e}_i, \bar{e}_j \rangle = \delta_{ij}$ otherwise. In this context, the rotations fix the line $\ell = \mathrm{span}\{\bar{e}_0\}$. We recall that in [37, p. 689], do Carmo and Dajczer obtained the general expression of the rotations fixing ℓ relative to the null frame, namely,

$$A_k = \begin{array}{cc} \begin{array}{ccccccccc} & & & & k & & & & \end{array} & \\ \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & * & \cdots & * & 0 & * & \cdots & * & 0 \\ \vdots & & & & \vdots & & & & \vdots \\ 0 & * & \cdots & * & 0 & * & \cdots & * & 0 \\ t_k & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & * & \cdots & * & 0 & * & \cdots & * & 0 \\ \vdots & & & & \vdots & & & & \vdots \\ 0 & * & \cdots & * & 0 & * & \cdots & * & 0 \\ -t_k^2/2 & 0 & \cdots & 0 & -t_k & 0 & \cdots & 0 & 1 \end{pmatrix} & \begin{matrix} \\ \\ \\ \\ \overset{k}{\leftarrow} \\ \\ \\ \\ \\ \end{matrix} \end{array}$$

where k is an integer between 1 and n and the entries marked with an asterisk (characterized in [37]) will not be used in our arguments below.

To give the explicit parametrization of a rotation hypersurface in this setting, we must consider two subcases, depending again on the character of the plane π containing the curve γ generating the hypersurface:

(a) if π is not null, then it contains two linearly independent null vectors. In fact, we may suppose that $\pi = \mathrm{span}\{\bar{e}_0, \bar{e}_{n+1}\}$. Define

$$(t_1, \ldots, t_n, s) \mapsto \left(x_0, x_0 t_1, \ldots, x_0 t_n, -\frac{T^2}{2} x_0 + x_{n+1} \right), \tag{15}$$

where $T^2 = \sum t_j^2$ and $\gamma(s) = (x_0(s), x_{n+1}(s))$ is a curve in the plane spanned by $\bar{e}_0$ and $\bar{e}_{n+1}$. In order to obtain an immersion from (15), we ask for $x_0 \neq 0$ as well as $(x_0')^2 + (x_{n+1}')^2 \neq 0$.

The determinant of the metric associated to the parametrization (15) is equal to $2(x_0^2)^n x_0' x_{n+1}'$, which under our assumptions vanishes only when x_{n+1} is constant. But in this case the hypersurface is contained in a light cone, as may be seen returning to the usual coordinates in $\mathbb{R}_1^{n+2}$. If $(y_0, \ldots, y_{n+1})$ are these coordinates, it is easily seen that they satisfy

$$\left(y_{n+1} - \frac{1}{\sqrt{2}} x_{n+1} \right)^2 = \sum_{i=1}^{n} y_i^2 + \left(y_n - \frac{1}{\sqrt{2}} x_{n+1} \right)^2,$$

which in turn means that the hypersurface is contained in a light cone with vertex in $x_{n+1}\bar{e}_0$;

(b) if π is null, say, $\pi = \mathrm{span}\{\bar{e}_0, \bar{e}_i\}$ for some $i \neq 0, n+1$, then the generating curve has the form $\gamma(s) = (x_0(s), x_i(s))$ relative to the null frame. From the matrix form A_k of the rotations in question, it is easy to see that each point in the orbit of γ does not have a component in the direction of $\bar{e}_{n+1}$; that is, the orbit lies entirely in the null hyperplane generated by $\bar{e}_0, \ldots, \bar{e}_n$ and hence the hypersurface is an open set of this hyperplane.

Since each null rotation hypersurface in $\mathbb{R}_1^{n+2}$ falls under one of the above cases, we have proved our theorem. $\square$

4. Totally umbilical hypersurfaces in the light cone

From now on we will study several questions related to the umbilicity in null hypersurfaces of the Minkowski space. Firstly, in the following proposition we generalize Example 2.4 characterizing the hypersurfaces of the light cone which are totally umbilical with respect to any normal vector field.

Proposition 4.1. *Let $S \subset \Lambda_0^{n+1}$ be a spacelike hypersurface of the light cone. S is U-totally umbilical with respect to any normal vector field U if and only if S is the intersection of the cone with a $(n+1)$-dimensional hyperplane not passing through the origin.*

Proof. Let $b = (b_0, \ldots, b_{n+1})$, $b \notin \mathrm{span}(\xi)$ be a fixed vector, and cut the light cone with a hyperplane orthogonal to b in order to obtain a hypersurface S of the light cone. At each point, the tangent space T_pS is spanned by

$$X = \sum_{i=0}^{n+1} a_i \partial_i, \quad \text{where } -a_0 b_0 + \sum_{i=1}^{n+1} a_i b_i = 0 \quad \text{and} \quad -a_0 u_0 + \sum_{i=1}^{n+1} a_i u_i = 0.$$

The null vector field η everywhere normal to S satisfying $\langle \eta, \xi \rangle = 1$ is given by

$$\eta = -\frac{1}{2\langle \xi, b \rangle^2} \xi + \frac{1}{\langle \xi, b \rangle} b$$

and using the fact that b is constant,

$$A_\eta X = -\frac{1}{2\langle \xi, b\rangle^2} X, \quad X \in \Gamma(TS),$$

so that S is η-umbilical and, as pointed out before, it is U-umbilical for each vector field U normal to S. The converse part of the proof follows the argument in Lemma 25, p. 73 of [39], which we adapt and outline here for completeness. Let S be U-umbilical relative to any normal vector field U. We already know that $A^*_\xi X = -X$. Let $A_\eta X = \lambda X$ for some function λ and every X tangent to S. On one hand, we use Codazzi equation in order to prove that λ is constant along each connected component of S. On the other hand, considering S as a spacelike hypersurface in $\mathbb{R}^{n+2}_1$ we may write the Weingarten formula

$$\widetilde{\nabla}_X \eta = -A_\eta X + \nabla^\perp_X \eta = -\lambda X + \nabla^\perp_X \eta.$$

By the usual properties of a connection, we have that $\langle \nabla^\perp_X \eta, \eta\rangle = 0$. Also, using the facts $\langle \xi, \eta\rangle = 1$ and $\nabla^\perp_X \xi = 0$ we obtain

$$0 = \langle \nabla^\perp_X \xi, \eta\rangle + \langle \xi, \nabla^\perp_X \eta\rangle = \langle \xi, \nabla^\perp_X \eta\rangle,$$

so that $\nabla^\perp_X \eta = 0$ as well and hence $\widetilde{\nabla}_X \eta = -\lambda X$. A standard calculation gives that the mean curvature vector of S in $\mathbb{R}^{n+2}_1$ is $H = \lambda\xi - \eta$; also recall that this vector satisfies the relation with the second fundamental form given by

$$h(X, Y) = \langle X, Y\rangle H.$$

Let us consider first the case $\lambda \neq 0$. We want to prove that the $(n + 1)$-dimensional distribution generated at each point by H and the vectors tangent to S is parallel, in order to invoke the results in [39]. Take then an orthonormal frame $e_1, \dots, e_n, e_{n+1}, e_{n+2}$ on $\mathbb{R}^{n+2}_1$ where the first n elements are everywhere tangent to S and e_{n+1} is the unit vector in the direction of H. Except for some (easy) calculations, we have to prove that for each X tangent to S and $i = 1, \dots, n$, $\widetilde{\nabla}_X e_i$ does not have a component in the direction of e_{n+2}; but

$$\langle \widetilde{\nabla}_X e_i, e_{n+2}\rangle = \langle h(X, e_i), e_{n+2}\rangle = \langle X, e_i\rangle\langle H, e_{n+2}\rangle = 0.$$

The above cited Lemma 25 states that a connected component of S lies in an $(n + 1)$-dimensional, totally geodesic submanifold of $\mathbb{R}^{n+2}_1$; namely, a hyperplane. On the other hand, $\lambda = 0$ coupled with $\nabla^\perp_X \eta = 0$ enables us to use Theorem 4.3 in [23] to assert that S lies in a null hyperplane, thus completing the proof. □

Remark 4.2. Theorem 4.1 in [40] gives a criterion to decide the U-umbilicity of a point, which can be directly extended to our setting: if a point p is U-umbilical then all normal vector fields V linearly independent from U have the same eigenvectors. For completeness we include the proof: if U_p, V_p are linearly independent, then each vector W_p normal to S at p and linearly independent from U_p may be written as $W = \lambda U + \mu V$ with $\mu \neq 0$ (we omitted the subindex p). Then

$$A_W X = \lambda A_U X + \mu A_V X = \lambda\kappa X + \mu A_V X,$$

since p is U-umbilical. From this expression it is easy to conclude that A_W and A_V share their eigenvectors.

Proposition 4.1 gives a characterization of the spacelike U-totally umbilical hypersurfaces of the light cone. We proceed now to exhibit non-totally umbilical examples. For this purpose we focus on a particular kind of surfaces $S \subset \Lambda^3_0 \subset \mathbb{R}^4_1$- namely, spherical rotation surfaces and find sufficient and necessary conditions for them to be non-totally umbilical.

Let

$$(x_0(u), x_1(u), x_2(u))$$

be a spacelike curve contained in $\Lambda^2_0 \subset \mathbb{R}^3_1$ parametrized by arc length; that is,

$$-x_0^2 + x_1^2 + x_2^2 = 0, \qquad -x_0'^2 + x_1'^2 + x_2'^2 = 1.$$

Hence S is parametrized by

$$\Phi(u, v) = (x_0(u), x_1(u), x_2(u)\cos v, x_2(u)\sin v).$$

The partial derivatives Φ_u, Φ_v are given by

$$\Phi_u = (x_0', x_1', x_2'\cos v, x_2'\sin v), \qquad \Phi_v = (0, 0, -x_2\sin v, x_2\cos v),$$

so that

$$E = \langle \Phi_u, \Phi_u\rangle = 1, \qquad F = \langle \Phi_u, \Phi_v\rangle = 0, \qquad G = \langle \Phi_v, \Phi_v\rangle = x_2^2.$$

If we suppose $x_2 \neq 0$, as we will do, then S is a spacelike surface contained in Λ_0^3. As usual, the position vector $\xi = \Phi$ is a null vector field normal to S. In order to find another null vector field η normal to S such that $\langle \xi, \eta \rangle = 1$, we observe that every vector V normal to S may be expressed as

$$V = \begin{vmatrix} -e_0 & e_1 & e_2 & e_3 \\ a_0 & a_1 & a_2 & a_3 \\ x_0' & x_1' & x_2' \cos v & x_2' \sin v \\ 0 & 0 & -x_2 \sin v & x_2 \cos v \end{vmatrix}$$

for a_0, a_1, a_2, a_3 scalars. Here e_0, e_1, e_2, e_3 is the canonical basis of $\mathbb{R}_1^4$. For example, if we take $a_0 = 1$ and $a_1 = a_2 = a_3 = 0$ and then divide by x_2 we obtain the normal vector field

$$V = (0, -x_2', x_1' \cos v, x_1' \sin v).$$

Since ξ and this V are linearly independent, the null vector η we are looking for is a linear combination of them:

$$\eta = a\xi + bV = (ax_0, ax_1 - bx_2', (ax_2 + bx_1') \cos v, (ax_2 + bx_1') \sin v).$$

Since η must satisfy $\langle \eta, \eta \rangle = 0$ and $\langle \eta, \xi \rangle = 1$, we must have

$$1 = \langle \eta, \xi \rangle = b\langle V, \xi \rangle, \qquad 0 = \langle \eta, \eta \rangle = 2ab\langle V, \xi \rangle + b^2\langle V, V \rangle = 2a + b^2\langle V, V \rangle,$$

and then

$$\eta = -\frac{\langle V, V \rangle}{2\langle V, \xi \rangle^2}\xi + \frac{1}{\langle V, \xi \rangle}V = -\frac{x_1'^2 + x_2'^2}{2(-x_1x_2' + x_1'x_2)^2}\xi + \frac{1}{-x_1x_2' + x_1'x_2}V.$$

We calculate the shape operator of S relative to η, obtaining

$$-A_\eta X = \frac{x_1'^2 + x_2'^2}{2(-x_1x_2' + x_1'x_2)^2}A_\xi^* X + \frac{1}{-x_1x_2' + x_1'x_2}P(\widetilde{\nabla}_X V)$$

for each $X \in \Gamma(TS)$. Here P denotes the projection onto TS, as in Section 2. Note that

$$\widetilde{\nabla}_{\Phi_v} V = V_v = (0, 0, -x_1' \sin v, x_1' \cos v) = \frac{x_1'}{x_2}\Phi_v, \tag{16}$$

meaning that Φ_v is a principal direction. Since the other principal direction must be orthogonal to this one, Φ_u must be that principal direction. In fact,

$$\widetilde{\nabla}_{\Phi_u} V = V_u = (0, -x_2'', x_1'' \cos v, x_1'' \sin v).$$

In order to obtain the projection V_u into TS, we express this vector as

$$V_u = A\Phi_u + B\Phi_v + C\xi + D\eta;$$

taking the scalar product of the above expression with Φ_u and Φ_v, we can see that

$$\langle V_u, \Phi_u \rangle = A\langle \Phi_u, \Phi_u \rangle, \qquad \langle V_u, \Phi_v \rangle = B\langle \Phi_v, \Phi_v \rangle,$$

but

$$\langle V_u, \Phi_u \rangle = -x_1'x_2'' + x_1''x_2', \qquad \langle V_u, \Phi_v \rangle = 0,$$

and then

$$P(V_u) = (-x_1'x_2'' + x_1''x_2')\Phi_u,$$

which proves that

$$-A_\eta \Phi_u = \frac{x_1'^2 + x_2'^2}{2(-x_1x_2' + x_1'x_2)^2}A_\xi^* \Phi_u + \frac{-x_1'x_2'' + x_1''x_2'}{-x_1x_2' + x_1'x_2}\Phi_u, \tag{17}$$

and Φ_u is a principal direction. We have just shown the following:

Proposition 4.3. *Let $\Phi(u, v) = (x_0(u), x_1(u), x_2(u) \cos v, x_2(u) \sin v)$ be the parametrization of a spacelike spherical rotation surface $S \subset \Lambda_0^3 \subset \mathbb{R}_1^4$. Then S is totally umbilical if and only if*

$$\frac{x_1'}{x_2} = -x_1'x_2'' + x_1''x_2'.$$

Since in general, we may find easily functions such that

$$\frac{x_1'}{x_2} \neq -x_1'x_2'' + x_1''x_2',$$

we may produce a large number of non-totally umbilical surfaces of the light cone. An interesting example of such a surface is given by

$$\Phi(u, v) = a(\cosh u, \sinh u, \cos v, \sin v), \quad a > 0. \tag{18}$$

Liu has shown that this is the only surface in the light cone Λ_0^3 that is homogeneous but non-totally umbilical [34]. In fact, we will characterize it as the only spacelike rotation minimal surface of Λ_0^3.

Theorem 4.4. *Let $S \subset \Lambda_0^3$ be a spacelike η-minimal rotation surface. Then S can be parametrized locally as $\Phi(u, v) = (\cosh u, \sinh u, \cos v, \sin v)$.*

Proof. We first note that due to Eqs. (16) and (17) the η principal curvatures of S are

$$k_u = -\frac{(x_1')^2 + (x_2')^2}{2(-x_1x_2' + x_1'x_2)^2} + \frac{x_1''x_2' - x_1'x_2''}{x_1'x_2 - x_1x_2'}$$

$$k_v = -\frac{(x_1')^2 + (x_2')^2}{2(-x_1x_2' + x_1'x_2)^2} + \frac{x_1'/x_2}{x_1'x_2 - x_1x_2'}$$

and hence, the minimality condition translates to

$$\frac{(x_1')^2 + (x_2')^2}{(-x_1x_2' + x_1'x_2)^2} = \frac{x_1''x_2' - x_1'x_2'' + x_1'/x_2}{x_1'x_2 - x_1x_2'}. \tag{19}$$

Since $S \subset \Lambda_0^3$ is spacelike we may define φ in such a way that $x_1 = x_0 \cos\varphi$ and $x_2 = x_0 \sin\varphi$. It follows at once that

$$x_0\varphi' = 1$$

and

$$\varphi = \int_{t_0}^{t} \frac{ds}{x_0(s)}.$$

Thus Eq. (19) reduces to

$$\frac{(x_0')^2 + 1}{x_0^2} = \frac{(x_0''x_0 - (x_0')^2 - 2)\sin\varphi + x_0'\cos\varphi}{-x_0^2 \sin\varphi},$$

or equivalently

$$(1 - x_0''x_0)\sin\varphi - x_0'\cos\varphi = 0. \tag{20}$$

We further differentiate the above equation to obtain an ordinary differential equation in x_0:

$$x_0'''x_0'x_0^2 + 3x_0''x_0 - 2(x_0'')^2x_0^2 + x_0''(x_0')^2x_0 - (x_0')^2 - 1 = 0. \tag{21}$$

As can be readily checked, the function

$$x_0^*(u) = \cosh(u - u_0)$$

is a solution to Eq. (21) and thus the surface given in Eq. (18) is indeed η-minimal.

Furthermore, if $x_0 = x_0(u)$ is a solution to (21) we can translate by means of an isometry the point $p_0 = (x_0(u_0), x_1(u_0), x_2(u_0)) \subset \Lambda_0^3$ to $p = (1, 0, 1)$. Thus $x_0(u_0) = 1$, $\varphi = \pi/2$ and $x_0''(u_0) = 1$ in virtue of Eq. (20). Moreover, by applying l'Hôpital rule to the relation

$$x_0' = \frac{1 - x_0''x_0}{\cot\varphi}$$

we obtain

$$x_0'(u_0) = \lim_{u \to u_0} \frac{-x_0'''x_0 - x_0''x_0'}{\varphi' \csc^2\varphi} = -x_0'''(u_0) - x_0'(u_0),$$

or equivalently,

$$x_0'''(u_0) = -2x_0'(u_0). \tag{22}$$

Finally, if we evaluate Eq. (21) at $u = u_0$ we find

$$x_0'''(u_0)x'(u_0) = 0. \tag{23}$$

From Eqs. (22) and (23) it then follows that

$$x_0'(u_0) = 0 = x_0'''(u_0).$$

We have then shown that x_0 satisfies the same initial conditions – up to third order – as the function $x_0^*(u) = \cosh(u - u_0)$. Since Eq. (21) satisfies the hypothesis of the fundamental theorem for the existence and uniqueness of solutions to implicit ordinary differential equations (see for instance Theorem 4.12, p. 164 and Hypothesis 4.2, p. 155 in [41]) then $x_0 \equiv x_0^*$ in a neighborhood of u_0 and the result follows. □

5. Principal configurations on spacelike surfaces immersed in the light cone of Minkowski 4-space

The existence of a non-totally umbilical spacelike surface $S \subset \Lambda_0^3 \subset \mathbb{R}_1^4$ enables us to pursue a systematic study of the geometry around an isolated umbilical point. In this section, we describe the differential equation of principal configurations around an isolated umbilical point relative to a null normal vector field η of a generic spacelike surface S immersed in the light cone Λ_0^3 of $\mathbb{R}_1^4$. As was observed in [33], the concept of principal curvature lines is derived from the existence of a self-adjoint operator with respect to a given metric and with real eigenvalues. Since A_η is $\Gamma(TS)$-valued and TS is integrable, Theorem 2.2.6 in [16] guarantees that the shape operator A_η restricted to TS is self-adjoint. Then, for each $p \in S$ there is an orthonormal basis of eigenvectors of A_η in T_pS with corresponding real eigenvalues since the metric on a spacelike surface is positive definite. These eigenvalues are called *η-principal curvatures* at p and, according to the definition of umbilicity given in Section 2, a point in S is *η-umbilical* if both η-principal curvatures coincide at that point. On the other hand, for any non-umbilical point there are two η-principal directions X that define two smooth line fields by the equation $A_\eta X = kX$ whose integral lines are called *η-principal curvature lines*.

An isolated η-umbilical point $p \in S$ together with the two families of η-principal curvature lines on the surface S around p form the local *η-principal configuration at p*. The equation $A_\eta(c'(t)) = k(t)c'(t)$ is the differential equation of the η-principal curvature lines $c(t)$, which may be obtained in a coordinate chart in the following way.

Let Φ be a parametrization of an open neighborhood $\mathcal{U} \subset S$ with local coordinates (x, y). For each $p = \Phi(x, y)$, the associated basis of T_pS is given by $\Phi_x = \partial\Phi/\partial x$ and $\Phi_y = \partial\Phi/\partial y$. In view of Eq. (7), we may define the *coefficients of the screen second fundamental form* as

$$\begin{aligned} e_\eta &= \langle \Phi_{xx}, \eta \rangle = \langle \Phi_x, A_\eta \Phi_x \rangle, \\ f_\eta &= \left\langle \Phi_{xy}, \eta \right\rangle = \langle \Phi_x, A_\eta \Phi_y \rangle = \langle \Phi_y, A_\eta \Phi_x \rangle, \\ g_\eta &= \left\langle \Phi_{yy}, \eta \right\rangle = \langle \Phi_y, A_\eta \Phi_y \rangle. \end{aligned}$$

By the elimination of the parameter k in the differential equation of the η-principal curvature lines we obtain in this chart

$$A\,(x, y)\;\; dy^2 + B\,(x, y)\;\; dx\,dy + C\,(x, y)\;\; dx^2 = 0,$$

where

$$\begin{aligned} A &= f_\eta G - g_\eta F, \\ B &= e_\eta G - g_\eta E, \\ C &= e_\eta F - f_\eta E, \end{aligned}$$

and $E,\ F,\ G$ are the coefficients of the first fundamental form of the immersion defined by Φ. In the pioneer works [27,28] the generic principal configurations around an isolated umbilical point with respect to the unique normal direction of a surface immersed in $\mathbb{R}^3$ with the usual Euclidean metric were determined by analyzing a differential equation which is similar to the differential equation obtained here for the η-principal curvature lines. The generic principal configurations named *Darbouxian* in [29] also appear in our context. We determine also the conditions which define an isolated η-umbilical point as a *simple* umbilical point (which is the first type of umbilical points one should consider in a study which pursues a classification of principal configurations in any context). In a forthcoming paper, we describe the topological type of Darbouxian principal configurations around simple η-umbilical points in spacelike surfaces immersed in the 3-dimensional light cone of $\mathbb{R}_1^4$ depending on some set of adequate parameters.

Let us begin our study with a generic spacelike surface S immersed in the light cone Λ_0^3 of $\mathbb{R}_1^4$. Without loss of generality we may suppose that the immersion Φ has the form

$$\Phi(x, y) = \left(\sqrt{x^2 + y^2 + f(x, y)^2}, x, y, f(x, y)\right),$$

where $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function whose graph $(x, y, f(x, y))$ in $\mathbb{R}^3$ is lifted to the light cone of $\mathbb{R}_1^4$. Moreover, modulo a rotation on the plane (x, y) and a dilation on the light cone, we may suppose that $f(0, 0) > 0$ and $f_y(0, 0) = 0$, so that f has the following 3-jet around the origin:

$$f(x, y) = f_o + f_{xo}x + \frac{1}{2}f_{xxo}x^2 + f_{xyo}xy + \frac{1}{2}f_{yyo}y^2 + \frac{a}{6}x^3 + \frac{d}{2}x^2y + \frac{b}{2}xy^2 + \frac{c}{6}y^3$$

where the subindex o in all functions means the evaluation at the origin. Consequently, the 3-jet of $g(x, y) = \sqrt{x^2 + y^2 + f(x, y)^2}$ around the origin becomes

$$g(x, y) = f_o + f_{xo}x + \frac{1}{2}\left(f_{xxo} + \frac{1}{f_o}\right)x^2 + f_{xyo}xy + \frac{1}{2}\left(f_{yyo} + \frac{1}{f_o}\right)y^2 + \frac{1}{6}\left(a - \frac{3f_{xo}}{f_o^2}\right)x^3 + \frac{d}{2}x^2y + \frac{1}{2}\left(b - \frac{f_{xo}}{f_o^2}\right)xy^2 + \frac{c}{6}y^3.$$

By Theorem 2.2.6 of [16], the shape operator $A_\eta : T_pS \to T_pS$ is self-adjoint, where η is a null normal vector of S at p. Let $p = \Phi(0, 0)$ be an isolated umbilical point of S with respect to η, which satisfies

$$\langle \eta, \eta \rangle_p = 0, \qquad \langle \eta, \Phi_x \rangle_p = 0, \qquad \langle \eta, \Phi_y \rangle_p = 0, \qquad \langle \eta, \Phi \rangle_p = 1.$$

These conditions give us the following value of η at the origin:

$$\eta(0, 0) = \left(\frac{-1 - f_{xo}^2}{2f_o}, -\frac{f_{xo}}{f_o}, 0, \frac{1 - f_{xo}^2}{2f_o}\right).$$

Now, the η-umbilicity at p implies that we can write the second order derivatives of $f(x, y)$ at the origin as

$$f_{xx}(0, 0) = f_{yy}(0, 0) = (2kf_o^2 - f_{xo}^2 - 1)/(2f_o),$$
$$f_{xy}(0, 0) = 0,$$

where k is the common value of the principal curvatures at p. The 3-jet of f becomes

$$f(x, y) = f_o + f_{xo}x + \alpha\left(x^2 + y^2\right) + \frac{a}{6}x^3 + \frac{d}{2}x^2y + \frac{b}{2}xy^2 + \frac{c}{6}y^3,$$

where

$$\alpha = \left(2kf_o^2 - f_{xo}^2 - 1\right)/4f_o,$$

and the 3-jet of $g(x, y)$ becomes

$$g(x, y) = f_o + f_{xo}x + \beta\left(x^2 + y^2\right) + \left(a - \frac{3f_{xo}}{f_o^2}\right)\frac{x^3}{6} + \frac{d\,x^2y}{2} + \left(b - \frac{f_{xo}}{f_o^2}\right)\frac{xy^2}{2} + \frac{cy^3}{6},$$

where

$$\beta = \left(2kf_o^2 - f_{xo}^2 + 1\right)/4f_o.$$

Straightforward computations with these 3-jets of f and g give us the 1-jet of the vector field η around p with components

$$\eta_0(x, y) = -\frac{1}{2f_o} - \frac{f_{xo}^2}{2f_o} + \left(\frac{2kf_{xo} + (f_{xo}^2 + 1)m}{f_{xo}^2 - 1}\right)x + \left(\frac{f_{xo}^2 + 1}{f_{xo}^2 - 1}\right)ny,$$
$$\eta_1(x, y) = -\frac{f_{xo}}{f_o} + \left(\frac{k(f_{xo}^2 + 1) + 2f_{xo}m}{f_{xo}^2 - 1}\right)x + \frac{2f_{xo}}{f_{xo}^2 - 1}ny,$$
$$\eta_2(x, y) = -ky,$$
$$\eta_3(x, y) = mx + ny + \frac{1}{2f_o} - \frac{f_{xo}^2}{2f_o},$$

where m, n are the arbitrary real numbers. With the 1-jet of η and the 3-jet of the immersion Φ we obtain the coefficients e_η, f_η, g_η of the screen second fundamental form. Using these and the coefficients of the first fundamental form E, F, G we obtain, after a standard procedure, the following.

Proposition 5.1. *The 1-jet of the differential equation of η-principal curvature lines for a generic spacelike surface S immersed in the light cone Λ_0^3 of $\mathbb{R}_1^4$ is given by*

$$A_1(x, y)\,dy^2 + B_1(x, y)\,dx\,dy - A_1(x, y)\,dx^2 = 0,$$

where

$$A_1(x, y) = \frac{dx + by}{f_o} + \frac{2\alpha f_{xo}y}{f_o^2}$$

and

$$B_1(x, y) = \frac{(a - b)x + (d - c)y}{f_o} + \frac{4\alpha f_{xo}x}{f_o^2}.$$

The types of principal configurations which appear generically are therefore Darbouxian.

In [28,27] it can be seen a description of Darbouxian types. Observe that A_1 and B_1 only depend on the 3-jet of the immersion Φ and not on the normal vector field η, a feature established in the case of surfaces immersed in Euclidean 3-space by Sotomayor and Gutierrez [27].

The η-umbilical points with Darbouxian principal configurations around them are special types of isolated *simple* η-umbilical points, which may be defined as the minimal nondegenerate points of the function

$$B_1^2(x, y) - 4A_1(x, y)C_1(x, y).$$

Straightforward computations give us the following.

Corollary 5.2. *The simple η-umbilical points of a generic spacelike surface S immersed in the light cone Λ_0^3 of $\mathbb{R}_1^4$ are determined, in the space of parameters $(f_o, f_{xo}, k, a, b, c, d)$, by $d \neq 0$ and k different from*

$$\frac{1 + f_{xo}}{2f_o^2} - \frac{a + b}{4f_{xo}} \pm \frac{1}{4f_{xo}} \left((a - 3b)^2 + 8d(d - c)\right)^{1/2}.$$

Consequently, the set of simple η-umbilical points is open in the space of parameters and its complement is a union of 5-manifolds.

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