

An Exercise of Combinatorics and an Application to Ditalgebras

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Abstract

Here we analyze a combinatoric exercise and one application to the theory of ditalgebras.

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1 Introduction

There are many interactions between several branches of Mathematics, and it is very common to apply combinatorics to solve problems in algebra.

We review an exercise that we proposed for a mathematical olympiad, and then we get back to its original setting, an indecomposable object in the category of modules of a ditalgebra, in order to compute a bound from the paper [2].

2 The exercise

Proposition 2.1 *Let there be n dots in the plane such that there are no three in the same line. Then draw line segments between pairs of points, such*

that any point is at the end of some of those line segments. Now associate to each point a positive integer (repetitions are allowed) and compute the product of the numbers of a pair of points if and only if there is a line segment joining these points. If s denotes the sum of all the products of above and t denotes the sum of all the integers associated to the points, then prove that

$$ns \geq (n - 1)t.$$

Proof: We proceed by induction on n .

The case $n = 1$ is easy.

Now let be $n = 2$ and P and Q the points with respective natural numbers a, b . Then $t = a + b$ and $s = ab$. Since $a, b \geq 1$ we get $b \geq \frac{b}{a}$ and $2b \geq 1 + \frac{b}{a}$, and so $\frac{n}{n-1}s = 2ab \geq a + b = t$.

Now assume that the claim is true for any set of $n \in \{1, 2, \dots, k - 1\}$ points fulfilling the hypothesis and set $n = k$ points as in the statement.

Let x be the biggest of the numbers associated to the points and let P be a point with number x . Let Q be an adjacent point to P and y the number associated to Q .

Now we choose a partition $\{U, V\}$ of the k points (R will denote a point different of P and different of Q):

1. $P \in U$ and $Q \in V$.
2. If there is not a path of line segments from R to P or to Q then we place R in V .
3. If there is a path of line segments from R to Q such that P is not contained in that path then we place R in V ; observe that any point in that path is in V .
4. If any path of line segments from R to Q contains P then we place R in U .

Let be $m_V = |V|$ and $m_U = |U|$.

Notice that both U and V satisfies the hypothesis if $m_V > 1$ and $m_U > 1$.

Let be s_U the sum of the products for each pair of points in U joined by a line segment and t_U the sum of all the numbers associated to points in U . In a similar way we define s_V and t_V .

We have to prove that

$$k(s_U + s_V + xy) \geq (k - 1)(t_U + t_V) \quad (1)$$

By the induction hypothesis we have $m_U s_U \geq (m_U - 1)t_U$ and $m_V s_V \geq (m_V - 1)t_V$, and so

$$\begin{aligned} \frac{k}{k-1}(s_U + s_V + xy) &\geq \frac{k}{k-1} \left(\frac{m_U-1}{m_U} t_U + \frac{m_V-1}{m_V} t_V + xy \right) \\ &= \frac{k(m_U-1)m_V t_U + k(m_V-1)m_U t_V + km_U m_V xy}{(k-1)m_U m_V}. \end{aligned}$$

Then it is enough to prove the inequality

$$k(m_U - 1)m_V t_U + k(m_V - 1)m_U t_V + km_U m_V xy \geq (k - 1)m_U m_V (t_U + t_V),$$

which is equivalent to prove $km_U m_V xy \geq m_V (k - m_U) t_U + m_U (k - m_V) t_V$. Since $k = m_U + m_V$ the previous inequality is equivalent to

$$m_U^2 m_V xy + m_U m_V^2 xy \geq m_V^2 t_U + m_U^2 t_V \quad (2)$$

Recall that x is the biggest number associated to the points, and so $t_V \leq |V|x = m_V x \leq m_V xy$, then it follows $m_U^2 t_V \leq m_U^2 m_V xy$. In a similar way we get $t_U \leq |U|x = m_U x \leq m_U xy$ and $m_V^2 t_U \leq m_U m_V^2 xy$. Adding up this last inequalities we get (2) and then follows (1). \diamond

Let us observe that in the graph $\cdot - \cdot - \dots - \cdot - \cdot$, where each point has associated the integer 1, the sum is n and the sum of the products is $n - 1$.

3 The application

Let k be a field, $R = D_1 \times \dots \times D_n$ where n is a natural number and each D_i is a division ring over a finite-dimensional k -algebra, W_0 and W_1 are finitely generated $R - R$ -bimodules.

The definitions of a ditalgebra \mathcal{A} and the category of finite-dimensional \mathcal{A} -modules, the last one denoted by $\mathcal{A}\text{-mod}$, can be consulted in [1] and [2]; we only need to recall the following facts:

1. \mathcal{A} is determined by the tensorial k -algebra $T = T_R(W) = R \oplus W \oplus W \otimes_R W \oplus \dots \oplus W^{\otimes n} \oplus \dots$, where $W = W_0 \oplus W_1$, and a differential $\delta : T \rightarrow T$ (see definition 1.6 of [1]).
2. If $M \in \mathcal{A}\text{-mod}$ then M is an A -module of finite dimension over k where A is the tensorial k -algebra $T_R(W_0) = R \oplus W_0 \oplus W_0 \otimes_R W_0 \oplus \dots \oplus W_0^{\otimes n} \oplus \dots$ (see definition 2.2 and remark 2.5 of [1]).
3. If M is an indecomposable \mathcal{A} -module then it has to be an indecomposable A -module (see definition 2.2 and remark 2.5 of [1]).

We will denote by e_j the idempotent of R given by $(0, \dots, 0, 1_j, 0, \dots, 0)$, where 1_j denotes the identity in D_j .

If M is a non-zero A -module then there exists a not empty subset $C = \{j_1, \dots, j_m\} \subset \{1, 2, \dots, n\}$ such that $e_{j_h} M \neq 0$ for each $j_h \in C$.

It is not hard to verify that if M is an indecomposable A -module and C is as above, then the graph with points in C and line segments determined by the non-zero $D_{j_h} - D_{j_{h'}}$ -bimodules $e_{j_h} W_0 e_{j_{h'}}$ is connected.

The ring $E_M = \text{End}_{\mathcal{A}}(M)^{op}$ acts in a natural way (see definition 2.3 of [2]) over each $e_{j_h}M$, with $j_h \in C$, and so we have a number d_{j_h} which is the length of $e_{j_h}M$ as right E_M -module.

By the claims above and the proposition 2.1 we get the following.

Proposition 3.1 *Let \mathcal{A} be a ditalgebra as above and $M \in \mathcal{A}\text{-mod}$ indecomposable. Let be C and d_{j_h} as above. Let s be the sum of the products $d_{j_h}d_{j_{h'}}$, where the $D_{j_h} - D_{j_{h'}}$ -bimodule $e_{j_h}W_0e_{j_{h'}}$ is different of zero.*

Then

$$ms \geq (m - 1) (d_{j_{h_1}} + \dots + d_{j_{h_m}}).$$

The propositions 2.1 and 3.1 can be used to obtain easily the inequality of proposition 4.10 of [2].

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