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Codimension one bifurcations of non simple umbilical points for surfaces immersed in $\mathbb{R}^4$

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Abstract

We study the qualitative change on the principal configurations around non simple umbilical points which are codimension one bifurcation points in the space $\mathcal{N}_1$ of 1-jets of vector fields ν normal to a generic surface immersed in $\mathbb{R}^4$ with an isolated ν-umbilical point. We prove that, except for one point and under some generic conditions, the principal configurations around this class of non simple umbilical points are of the same topological type D_{23} which represent a bifurcation between the Darbouxian types D_2 and D_3 in the space $\mathcal{N}_1$.

Keywords and phrases : immersed surfaces, umbilical point, principal curvature lines, bifurcation of umbilics.

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1 Introduction

It is known that for smooth surfaces in 3-Euclidean space the unit normal vector in every point of the surface is uniquely determined modulo orientation. This is no longer true if the codimension of the surface is greather than one, as for immersions of the surface in $\mathbb{R}^4$, where the normal vector field lives in a two dimensional space. The fact that the shape operator of a surface depends on the normal vector field gives rise to the problem of the topological classification and dynamics of principal configurations around umbilical points depending of the normal vector field. The principal configuration is the foliation determined by the families of maximal and minimal principal curvature lines, with the umbilical points as singularities of the foliation. The study of principal configurations goes back to the work of Monge [14] and Darboux [3]. It was Darboux who established that there are three generic patterns, called now *Darbouxian umbilics* D_1, D_2 and D_3. More recently, Sotomayor and Gutierrez, applying techniqes of dynamical systems, showed in [20] that there exists a class of surfaces structurally stable under small C^3 deformations whose umbilic points are all Darbouxian. The topological type of Darbouxian umbilics is described in detail in [20]. See also [2]. There has been a plenty of generalizations of the topic in many ways. See for example [1], [2], [6], [8], [15], [18]. We mention specially the work [12] where the simplest patterns of principal configurations that appear generically on 1-parameter families of surfaces in $\mathbb{R}^3$ has been realized, given the complete analysis of codimension one umbilic bifurcations for this kind of surfaces. Codimension two umbilics in $\mathbb{R}^3$ are studied in [4]. For more on curvature lines of surfaces in Euclidean spaces and their history, see [5] and references therein. In [16] it was obtained the bifurcation diagram of principal configurations of a generic class of local surfaces immersed in $\mathbb{R}^4$ which have an isolated simple umbilical point with respect to a generic normal vector field. Later, in [17] it was considered the non generic cases of this class of umbilics, showing how these bifurcation diagrams changes depending on the surface and the normal vector field. In this paper we pursue the study done in [16] and [17], analyzing the non simple umbilical points that represent bifurcation points of codimension one. For the sake of completeness we give in Sections 2 and 3 the relevant preliminary facts about principal configurations for local surfaces M immersed in $\mathbb{R}^4$ around an isolated simple umbilical point $x \in M$ with respect to a normal vector field ν defined in a neighbourhood of x. Finally, in Section 4 we prove that, except for one point and under some generic conditions, the principal configurations of non simple ν-umbilical points of codimension one, named D_{23} in this paper, are topologically equivalent to the type $D^1_{2,3}$ defined in [12], and we show that it represents a codimension one bifurcation point in our setting, using techniques developed in [12] for surfaces in $\mathbb{R}^3$.

2 Principal configurations for surfaces in $\mathbb{R}^4$

Let M be an oriented surface immersed in $\mathbb{R}^4$ with the induced Euclidean metric. For each $x \in M$ consider the decomposition $T_x\mathbb{R}^4 = T_xM \oplus N_xM$, where N_xM is the orthogonal complement of T_xM in $\mathbb{R}^4$. Let $\bar{\nabla}$ be the Riemannian connection of $\mathbb{R}^4$. For $x \in M$ and a non zero vector $\nu \in N_xM$, the shape operator is the self-adjoint linear map defined by

$$S_\nu : T_xM \to T_xM, \quad S_\nu(X) = -(\bar{\nabla}_{\bar{X}}\bar{\nu})^\top,$$

where $\bar{\nu}$ and $\bar{X}$ are local smooth extensions to $\mathbb{R}^4$ of normal vector ν and tangent vector X respectively, $\top$ means the tangent component. The second fundamental form associated to S_ν is given by

$$II_\nu(X) = \langle S_\nu(X), X\rangle.$$

Now, for each $x \in M$ there exists an orthonormal basis of eigenvectors of S_ν in T_xM with real eigenvalues, namely, the *ν-principal curvatures*, which coincide with the extreme values of the second fundamental form on the unitary circle of T_xM. The point $x \in M$ is named *ν-umbilical* if the ν-principal curvatures coincide at x. Let $\mathcal{U}_\nu$ be the set of ν-umbilical points in M. For each $x \in M \setminus \mathcal{U}_\nu$ there exist two fields of directions defined by the eigenvectors of S_ν which are smooth. The corresponding orthogonal families $\mathcal{L}_\nu,\ l_\nu$ of integral curves are named *ν-principal curvature lines*. The triple $(\mathcal{U}_\nu, \mathcal{L}_\nu, l_\nu)$ is the *ν-principal configuration* of M. The curve $c : (-\delta, \delta) \to M$ is a ν-principal curvature line if and only if

$$S_\nu(c'(t)) = \lambda(c(t))c'(t), \tag{2.1}$$

where $\lambda(c(t))$ is a real-valued function defined on $c(t)$. For the local analysis of the ν-principal configuration, let (x_1, x_2, x_3, x_4) be a system of coordinates such that the ν-umbilical point x of the surface M is located at the origin of $\mathbb{R}^4$, the tangent plane at x is the (x_1, x_2)-plane and the normal vector field ν coincide with $(0, 0, 1, 0)$ at the origin. Then M can be parametrized in a neighbourhood of x by

$$X(u, v) = (u, v, \varphi(u, v), \psi(u, v)), \tag{2.2}$$

with

$$\begin{aligned} \varphi(u,v) = & \quad \frac{k}{2}(u^2+v^2) + \frac{a}{6}u^3 + \frac{d}{2}u^2v + \frac{b}{2}uv^2 + \frac{c}{6}v^3 + \mathcal{O}(4), \\ \psi(u,v) = & \quad \frac{\alpha}{2}u^2 + \frac{\gamma}{2}v^2 + \frac{\delta}{6}u^3 + \frac{\epsilon}{2}u^2v + \frac{\zeta}{2}uv^2 + \frac{\eta}{6}v^3 + \mathcal{O}(4), \end{aligned}$$

where k is the value of the principal curvatures at p and $a, b, c, d, \alpha, \gamma, \delta, \epsilon, \zeta, \eta \in \mathbb{R}$. The uv term of ψ has been eliminated by a suitable rotation of the (x_1, x_2)-plane. In coordinates (u, v) the equation (2.1) can be written

$$(f_\nu G - g_\nu F)\ dv^2 + (e_\nu G - g_\nu E)\ dv\, du + (e_\nu F - f_\nu E)\ du^2 = 0, \tag{2.3}$$

where

$$E = \langle X_u, X_u\rangle, \ \ F = \langle X_u, X_v\rangle, \ \ G = \langle X_v, X_v\rangle$$

and

$$e_\nu = \langle X_{uu}, \nu\rangle, \ \ f_\nu = \langle X_{uv}, \nu\rangle, \ \ g_\nu = \langle X_{vv}, \nu\rangle$$

are the coefficients of the first and second fundamental forms, respectively.

Assuming that ν defines an isolated umbilic at the origin, a direct computation shows that its 1-jet can be written

$$\nu_1(u,v) = (-ku, -kv, 1, mu + nv), \tag{2.4}$$

where $m, n \in \mathbb{R}$ [16]. We call *umbilical* the normal vector field ν given by (2.4). It is not difficult to prove that for any $(m,n) \in \mathbb{R}^2$ there is a normal vector field ν with its first jet as above [18]. Then, the differential equation (2.3) can be written as

$$A(u,v)\ dv^2 + B(u,v)\ du\,dv + C(u,v)\ du^2 = 0, \tag{2.5}$$

where the functions A, B, C have the following 2-jets:

$$\begin{aligned}
A_2(u,v) &= du + bv - k^3uv - k\alpha\gamma uv + m\epsilon u^2 + n\epsilon uv + m\zeta uv + n\zeta v^2,\\
B_2(u,v) &= (a - b + m(\alpha - \gamma))u + (-k^3 - k\alpha^2 + m(\delta - \zeta))u^2\\
&\quad + (d - c + n(\alpha - \gamma))v + (n(\delta - \zeta) + m(\epsilon - \eta))uv\\
&\quad + (k^3 + k\gamma^2 + n(\epsilon - \eta))v^2,\\
C_2(u,v) &= -A_2(u,v).
\end{aligned}$$

These expressions are obtained from the calculation of the coefficients of the first and second fundamental forms in the chart (u,v) and substituting into equation (2.3). On the other hand, let $\mathcal{I} : M \to \mathbb{R}^4$ be a C^∞ immersion of the local surface M and consider the projective line bundle $P\mathcal{I}(M)$ over $\mathcal{I}(M)$ with projection Π. Let $\mathcal{I}(x)$ be a ν-umbilical point in $\mathcal{I}(M) \subset \mathbb{R}^4$. Then there is a neighborhood $V(\mathcal{I}(x))$ with canonical parametrization (2.2) identifying $\mathcal{I}(x)$ with the origin. The two coordinate charts

$$(u, v; q = du/dv) \ \text{ and } \ (u, v; p = dv/du)$$

cover $\Pi^{-1}(V(\mathcal{I}(x)))$. The equation (2.5) defines in $P\mathcal{I}(M)$ a surface

$$S(\mathcal{I}, \nu) = \mathcal{F}^{-1}(0), \tag{2.6}$$

where $\mathcal{F} : P\mathcal{I}(M) \to \mathbb{R}$ is given by

$$\mathcal{F}(u,v,p) = A(u,v)\ p^2 + B(u,v)\ p + C(u,v). \tag{2.7}$$

Now, let $\mathcal{U}_\mathcal{I}$ be the set of ν-umbilical points of $\mathcal{I}(M)$. Away from $\Pi^{-1}(\mathcal{U}_\mathcal{I})$ the set (2.6) is a regular surface of $P\mathcal{I}(M)$. In fact it is a double covering of $\mathcal{I}(M) - \mathcal{U}_\mathcal{I}$ and assuming that the origin of $\mathbb{R}^4$ is an isolated ν-umbilical point, the real projective line $\Pi^{-1}(0,0)$ is contained in $S(\mathcal{I}, \nu)$. The set of umbilic points is the common locus of the curves $A(u,v) = 0, B(u,v) = 0$.

Condition(T). The pair $(\mathcal{I}, \nu)$ satisfies the *transversality condition* at (u_0, v_0) if the curves $A(u,v) = 0$ and $B(u,v) = 0$ intersect transversally at (u_0, v_0), with A and B defined in (2.5).

It is easy to see that condition (**T**) is equivalent to the regularity of $S(\mathcal{I}, \nu)$ along $\Pi^{-1}(0,0)$. Denoting by $\mathcal{F}_u$, $\mathcal{F}_v$ and $\mathcal{F}_p$ the partial derivatives with respect to u, v and p, the *Lie-Cartan* vector field of the pair $(\mathcal{I}, \nu)$ is defined by

$$\mathcal{X} = \mathcal{F}_p \frac{\partial}{\partial u} + p\mathcal{F}_p \frac{\partial}{\partial v} - (\mathcal{F}_u + p\mathcal{F}_v) \frac{\partial}{\partial p} \tag{2.8}$$

and have the following properties [16]:

(i) $\mathcal{X}$ is tangent to $S(\mathcal{I}, \nu)$,

(ii) $\Pi_*(\mathcal{X})$ vanishes only at the origin,

(iii) if $(u, v; p) \in S(\mathcal{I}, \nu)$ with $(u,v) \neq (0,0)$, then $\Pi_*(\mathcal{X}(u,v;p))$ generates the ν-principal curvature line with direction $(1,p)$,

(iv) the eigenvalues of the linear part of $\mathcal{X}$ at $(0,0,p)$ are

$$\begin{aligned}
&\beta_1 = 0,\\
&\beta_2 = 2b - a - (\alpha-\gamma)m + 2(c - 2d - (\alpha-\gamma)n)p - 3bp^2, \qquad (2.9)\\
&\beta_3 = a - b + (\alpha-\gamma)m + (3d - c + (\alpha-\gamma)n)p + 2bp^2.
\end{aligned}$$

On the other hand, since $\mathcal{F}_p(0,0,p)=0$ for any p, the singularities of $\mathcal{X}$ in $\Pi^{-1}(0,0)$ are the roots of the polynomial

$$f(p) = (\mathcal{F}_u + p\mathcal{F}_v)(0,0,p), \qquad (2.10)$$

which is called the *separatrix polynomial* of the pair $(\mathcal{I},\nu)$. The roots of (2.10) are the tools to describe the ν-principal configuration around an isolated ν-umbilical point. In fact, for the Darbouxian ν-umbilical points defined below, these roots are in direct correspondence with the tangent directions of the ν-principal curvature lines which contain the ν-umbilical point in their closure and the number of these directions is determined by the sign of the discriminant of (2.10) [18]. With the 1-jets of $A(u,v)$ and $B(u,v)$, the 1-jet of $\mathcal{F}$ can be written

$$\begin{aligned}
\mathcal{F}(u,v,p) = {} & (du+bv)p^2\\
& + ((a-b+m(\alpha-\gamma))u + (d-c+n(\alpha-\gamma))v)p\\
& - (du+bv), \qquad (2.11)
\end{aligned}$$

and the corresponding separatrix polynomial is given by

$$f(p) = bp^3 + (2d - c + (\alpha-\gamma)n)p^2 + (a - 2b + (\alpha-\gamma)m)p - d. \qquad (2.12)$$

Then, the eigenvectors associated to the eigenvalues (2.9) for each root $p_i, i=1,2,3$ of $f(p)$ become

$$\begin{aligned}
&\xi_1 = ((c - d - 2bp_i + n(\gamma-\alpha))/(a - b + 2dp_i + m(\alpha-\gamma)), 1, 0),\\
&\xi_2 = (1, p_i, 0),\\
&\xi_3 = (0,0,1).
\end{aligned}$$

The *simple* ν-umbilical points are defined as follows.

Definition 2.1. *Let x be an isolated ν-umbilical point of the surface (2.2) with ν-principal curvature lines around x defined by (2.5). We call x a **simple** ν-umbilical point if it is a non degenerate minimum of the function*

$$B^2(u,v) - 4A(u,v)C(u,v).$$

In terms of (2.2), (2.4) and the 1-jet of (2.5) the simple ν-umbilical points of M are characterized by the following two inequalities:

$$4d^2 + (a - b + (\alpha-\gamma)m)^2 > 0, \qquad (2.13)$$

$$((a-b)b + (c-d)d + (\alpha-\gamma)(bm - dn))^2 > 0, \qquad (2.14)$$

and they are classified according to the roots of the separatrix polynomial as follows.

Definition 2.2. *Let x be a simple ν-umbilical point of a surface M immersed in $\mathbb{R}^4$ with separatrix polynomial $f(p)$ associated to $(\mathcal{I},\nu)$, where $\mathcal{I}: M \to \mathbb{R}^4$ is the immersion at x. Then x is called:*

1. ***Darbouxian*** *if $f(p)$ has only simple roots and condition **(T)** holds.*
2. $\mathbf{D_{12}}$ *if $f(p)$ has one simple and one double root.*
3. $\tilde{\mathbf{D}}_{\mathbf{1}}$ *if $f(p)$ has a triple root.*

In [18] it was proved that the class of Darbouxian ν-umbilical points have local principal configurations with topological type stable in the sense that they are preserved under small perturbations of the pair $(\mathcal{I}, \nu)$ and are generic in the space of pairs $(\mathcal{I}, \nu)$ with the C^3-topology, under suitable conditions. These are generalizations to $\mathbb{R}^4$ of results obtained by Sotomayor and Gutierrez in the pioneering work [20], where they established the stability and genericity, in some sense, of Darbouxian principal configurations for surfaces immersed in $\mathbb{R}^3$. The Darbouxian ν-principal configurations are classified into three topological types depending on the roots of the separatrix polynomial (2.10) and the eigenvalues (2.9) as follows.

Definition 2.3. *A Darbouxian ν-umbilical point x of $(\mathcal{I}, \nu)$ is named:*

$\mathbf{D_1}$*: if $f(p)$ has only one real root p_1 and $\beta_2(p_1)\beta_3(p_1) < 0$,*

$\mathbf{D_2}$*: if $f(p)$ has three distinct real roots p_i with $p_1 < p_2 < p_3$ and $\beta_2(p_2)\beta_3(p_2) > 0$ while $\beta_2\beta_3 < 0$ for the other roots,*

$\mathbf{D_3}$*: if $f(p)$ has three distinct real roots p_i and $\beta_2\beta_3 < 0$ for all roots.*

These conditions determine the qualitative behavior of the ν-principal configuration around the ν-umbilical point in the following way. The properties (i) to (iii) of the Lie-Cartan vector field (2.8) can be used to obtain a blow up of the ν-umbilical point with $\mathcal{X}$ tangent to the pull back of the ν-principal curvature lines.

For D_1 we have only one singularity of $\mathcal{X}$ at $(0, 0, p_1)$ of saddle type. For D_2 we have three singularities along the projective line, one node between two saddles, and for D_3 we have three saddles along the projective line.

The projection to the surface M of the phase portraits in $S(\mathcal{I}, \nu)$ around these singularities of $\mathcal{X}$ produces the Darbouxian ν-principal configurations described in [18] for surfaces in $\mathbb{R}^4$ and are the same topological types defined and illustrated in [20] for surfaces in $\mathbb{R}^3$.

The type D_{12} is obtained by varying the conditions of D_2 until one of its saddle singularities coincides with the node singularity of a D_1 type, arising a saddle-node corresponding to the double root of the separatrix polynomial. The type D_{12} was characterized in [12], where it is named D_2^1, for surfaces immersed in $\mathbb{R}^3$ as a generic bifurcation of codimension one due to the violation, in the mildest possible way, of the Darbouxian condition while condition $(\mathbf{T})$ is preserved.

The fact that D_{12} umbilic points are codimension one bifurcation points was also proved in [16] for surfaces in $\mathbb{R}^4$ under some generic conditions. On the other hand, when the condition $(\mathbf{T})$ is violated another bifurcation point of codimension one appears for surfaces in $\mathbb{R}^3$, named $D_{2,3}^1$ in [12], where its principal configuration and bifurcation analysis was determined.

Finally, the type $\tilde{D}_1$ is obtained by varying the conditions for D_{12} until one saddle and one saddle-node coincide, arising a unique saddle in the projective line of $S(\mathcal{I}, \nu)$ corresponding to the triple root of the separatrix polynomial. The ν-principal configuration of the $\tilde{D}_1$ umbilical point is topologically equivalent to the Darbouxian type D_1 and represents a codimension two bifurcation point, as was showed in [16] for surfaces in $\mathbb{R}^4$ and also in [4] for surfaces in $\mathbb{R}^3$, named there D_1^2, where the superscript stands for the codimension and the subscript stands for the number of separatrices approaching the umbilical point. See Figure 1.

In the more general context of positive quadratic differential forms, Gutierrez and Guíñez in [11] characterized the types D_{12} and $\tilde{D}_1$ as the non locally stable simple singular points.

3 Simple umbilical points

Consider the local surface M parametrized by (2.2) and the normal vector field ν parametrized by (2.4). For any value of the ν-principal curvature k at the ν-umbilical point x and generic values of the parameters $a, d, b, c, \alpha, \gamma, \delta, \epsilon, \zeta, \eta$ in (2.2) and generic values of m, n in (2.4), the point $x \in M$ remains to be an isolated

ν-umbilical point. Let $\mathcal{N}_1$ denote the space of the 1-jets of this class of generic normal vector fields ν. The normal form of $\nu \in \mathcal{N}_1$ given by (2.4) allows us to identify $\mathcal{N}_1$ with $\mathbb{R}^2$ and coordinates (m, n). Let us recall that the simple ν-umbilical points of M are characterized by the inequalities (2.13) and (2.14). Observe that for $d \neq 0$ the inequality (2.13) is satisfied for all $\nu \in \mathcal{N}_1$. Consequently, to determine the set of non simple ν-umbilical points in $\mathcal{N}_1$ only remains to see where the second inequality (2.14) is not satisfied, and this occurs for the points (m, n) on the line L given by

$$dn = bm + \frac{(a-b)b + (c-d)d}{\alpha - \gamma}, \quad \alpha \neq \gamma. \tag{3.1}$$

If $\alpha = \gamma$ it follows from (2.2) that the point $x \in M$ is ν-umbilical with respect to the two linearly independent local normal vector fields which coincide with $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ at the origin and therefore, by the lineality of the shape operator with respect to ν, the point x is ν-umbilical for all $\nu \in N_xM$ which are not of interest in our study. Then we are going to assume from now that $\alpha \neq \gamma$. In [16] it was proved that for $\nu \in \mathcal{N}_1$ with coordinates (m, n) and $bd \neq 0$, the bifurcation set of the isolated simple ν-umbilical points in the space $\mathcal{N}_1$ is the line (3.1) and a real algebraic curve Γ which can be written, after one translation of coordinates, in the form

$$f_2(x, y) + f_3(x, y) + f_4(x, y) = 0,$$

where

$$\begin{aligned}
f_2(x, y) &= 27\,(bd)^{2/3}\left(\sqrt[3]{b}\;x + \sqrt[3]{d}\;y\right)^2, \\
f_3(x, y) &= 2(\alpha - \gamma)\left(2\left(\sqrt[3]{b}\;x - \sqrt[3]{d}\;y\right)^3 + 9\,(bd)^{1/3}\left(\sqrt[3]{b}\;x - \sqrt[3]{d}\;y\right)xy\right), \\
f_4(x, y) &= -(\gamma - \alpha)^2 x^2 y^2.
\end{aligned}$$

Consequently, the relevant parameters which determine the algebraic curve Γ are b and d for any α not equal to γ. This and the facts, proved in [16], that for generic given values of a, c, the line (3.1) have a unique intersection point with the curve Γ and it is a tangency point implies that the bifurcation set in the space $\mathcal{N}_1$ is generically determined by the parameters b, d. For $bd \neq 0$ the curve Γ has two connected components with one of them having a singular cusp point V corresponding to the type $\tilde{D}_1$ and the other one diffeomorphic to a line [16]. See Figure 1. Furthermore, all the regular points of Γ are of type D_{12}, except at the unique tangency point L_T with the line L given by

$$L_T = \frac{1}{\gamma - \alpha}\left(\frac{ab - d^2}{b}, \frac{b^2 - cd}{d}\right). \tag{3.2}$$

The main result of [16] is the bifurcation diagram of simple ν-umbilical points in the space $\mathcal{N}_1$ for $bd \neq 0$. The curve Γ and the line L divide the space $\mathcal{N}_1$ into five regions of Darbouxian ν-umbilical points distributed as shown in Figure 1. It remains to find the topological type of ν-principal configurations of the non simple ν-umbilical points. These are the points of the line L. Our main result here is that all points of L different from the tangency point L_T are of the same topological type D_{23} and are codimension one bifurcation points. For the point L_T the study of its ν-principal configuration and bifurcation analysis involve some technical work due to the appearance of a zero eigenvalue of multiplicity 3 of the linear part of (2.8) at one of its singularities on the projective line. In a forthcoming paper we pursue that study.

The non generic cases in which only one of the parameters b or d is equal to zero were described in [17], obtaining that the bifurcation diagram of simple ν-umbilical points in the space $\mathcal{N}_1$ has a behavior depending on the parameters as follows. Let $\nu \in \mathcal{N}_1$ and $a,\ c,\ \alpha,\ \gamma \in \mathbb{R}$ fixed, with generic values $\gamma > \alpha,\ a > 0,\ c > 0$. Then, the bifurcation diagram for $bd \neq 0$ is topologically equivalent to the one showed in Figure 1. The equivalence between these bifurcation diagrams is determined by a rotation of the double tangent line of Γ at its singular point V. The equation of this double tangent line is

$$\sqrt[3]{b}\;x + \sqrt[3]{d}\;y = 0. \tag{3.3}$$

For $d = 0$ and $b \neq 0$ the bifurcation set is the union of the parabola defined by

$$m + \frac{a - 2b}{\alpha - \gamma} = \frac{\alpha - \gamma}{4b}\left(n - \frac{c}{\alpha - \gamma}\right)^2$$

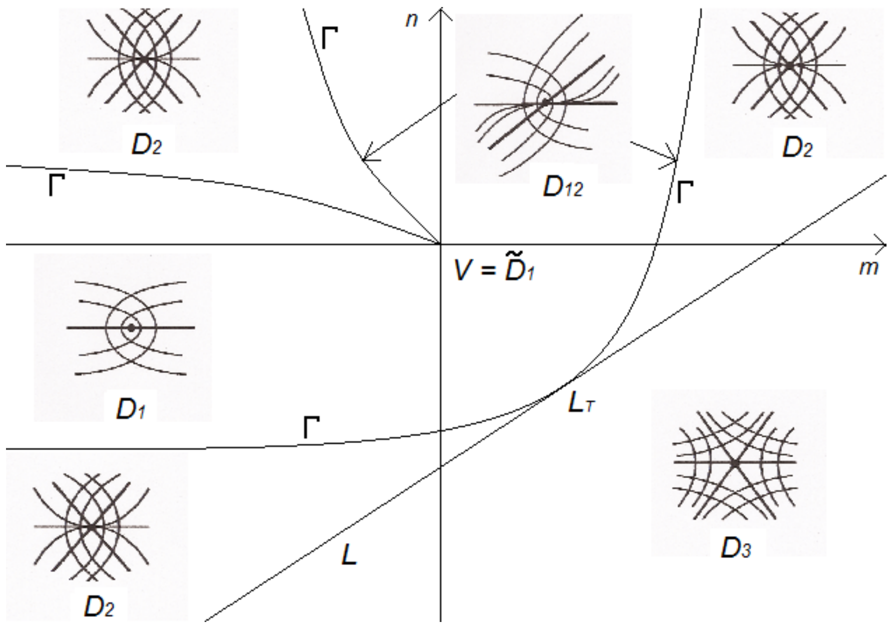

Figure 1: Bifurcation diagram in the space $\mathcal{N}_1$.

and the line

$$(\gamma-\alpha)\,m = a-b.$$

The points of the parabola different from its vertex are of type D_{12}, and the vertex is of type $\tilde{D}_1$. In this case the bifurcation diagram for $b>0$ and $\gamma>\alpha$ is the one shown in Figure 2 over the positive b-axis. When $d\neq 0$ and $b=0$ the bifurcation diagram is topologically equivalent to the previous case, depending only on the slope of the axis of the parabola, as showed in Figure 2. The curve Γ depends continuously on the parameters (b,d) and have limits which coincide with the parabola and line for the non generic cases just described when d or b approaches zero, but not both. The dependence of Γ on the parameters (b,d) is described in Figure 2 [17]. Finally, for $b=d=0$ all parallel lines to the coordinate axis are ν-principal curvature lines and there are no isolated ν-umbilical points, which are out of our consideration.

4 Non simple umbilical points of codimension one

In this Section we are going to describe the local bifurcation pattern that occurs at the non simple ν-umbilical points of codimension one in the space $\mathcal{N}_1$. In Section 3 we have seen that the locus of the non simple points is the line L gived by (3.1). On the other hand the transversality condition (**T**) at the ν-umbilical point of (2.2) happens when the tangent lines at $(0,0)$ to the curves $A(u,v)=0$ and $B(u,v)=0$ are not parallel, which means that

$$d(-c+d+n(\alpha-\gamma)) \neq b(a-b+m(\alpha-\gamma)). \tag{4.1}$$

Comparing this with (3.1) we see that the transversality condition (**T**) is violated on every point of the line L. Therefore, the transversality condition has to be discarded in order to obtain non simple ν-umbilical points. By introducing the equation (3.1) on (2.11) the corresponding separatrix polynomial becomes

$$f(p) = (d+bp)\left(dp^2+p(a-b+m(\alpha-\gamma))-d\right) \tag{4.2}$$

with roots

$$\begin{aligned}
p_1 &= -d/b,\\
p_2 &= \left(b-a-m(\alpha-\gamma)-\sqrt{4d^2+(a-b+m(\alpha-\gamma))^2}\right)/2d,\\
p_3 &= \left(b-a-m(\alpha-\gamma)+\sqrt{4d^2+(a-b+m(\alpha-\gamma))^2}\right)/2d,
\end{aligned}$$

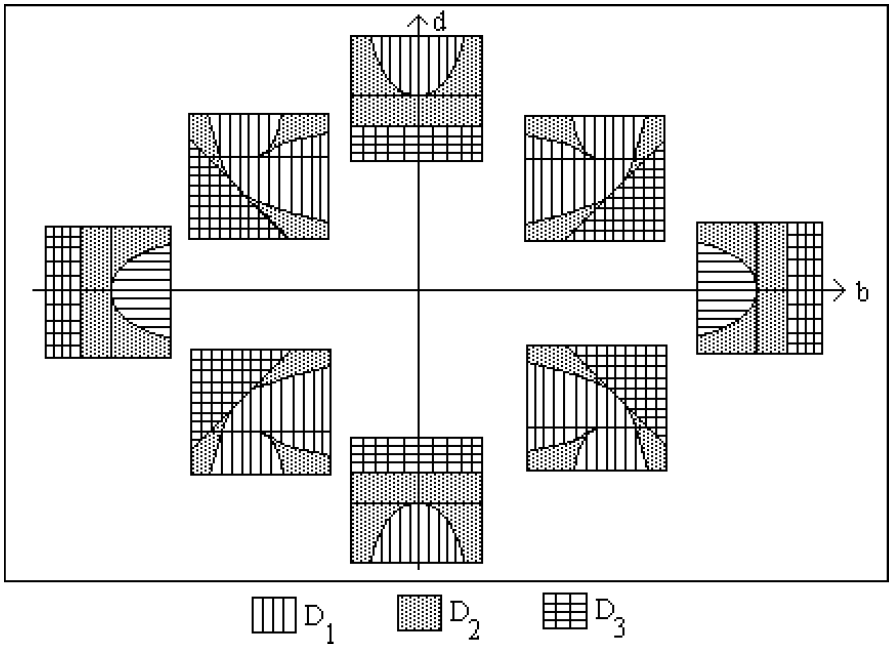

Figure 2: Bifurcation diagram depending on parameters (b, d) of the surface and (m, n) of the normal vector field.

which are well defined if $bd \neq 0$. At the point L_T where Γ and L are tangent, the polynomial (4.2) reduces to

$$f(p) = (d + bp)^2(b - dp) \tag{4.3}$$

with the double root $p = -d/b$ and the corresponding linear part of the Lie-Cartan vector field (2.8) have a zero eigenvalue of multiplicity 3 at $(0, 0, -d/b)$. That kind of multiplicity for a zero eigenvalue of the linear part of the Lie-Cartan vector field occurs in only one of the four topological types of codimension two umbilics for surfaces in $\mathbb{R}^3$, classified by Garcia and Sotomayor in [4]. It is named D^2_{2h} by them. The other three codimension two umbilics in $\mathbb{R}^3$ are: D^2_1, which correspond to our point $\tilde{D}_1$ in the space $\mathcal{N}_1$; D^2_{2p}, which is topologically a D_2 umbilic; and D^2_3, which is topologically a D_3 umbilic. Here we restrict our analysis to the points of L different from L_T. Following the method developed in [12] we are going to show in Theorem 4.2 that every point in $L - \{L_T\}$ is topologically equivalent to an umbilic of the type $D^1_{2,3}$ described there. See Figure 3.

Definition 4.1. *The ν-umbilical point of (2.2) is of type* $\mathbf{D_{23}}$ *if the transversality condition (**T**) fails at two singularities of the Lie-Cartan vector field (2.8) over the projective line $\Pi^{-1}(0, 0)$ at which the surface (2.6) have non degenerate singularities of Morse type.*

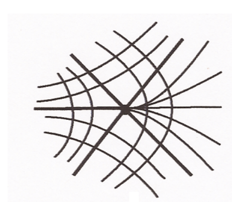

Figure 3: Topological type D_{23}.

Theorem 4.2. *The ν-umbilical points of $L - \{L_T\}$ in the space $\mathcal{N}_1$ are of type D_{23} for $bd \neq 0$ and $\Delta \neq 0$, where Δ is the Hessian determinant of the 2-jet of (2.7) at the two singularities of the Lie-Cartan vector field (2.8) described in Definition (4.1).*

Proof. To simplify the calculations let us begin with the translation

$$p \mapsto p + p_1,$$

where $p_1 = -d/b$ is the first root of the separatrix polynomial (4.2). After this translation the vector field $\mathcal{X}(u,v,p)$ has linear part $D\mathcal{X}$ with the following eigenvalues at the origin:

$$\lambda_1 = 0, \quad \lambda_2 = 0, \quad \lambda_3 = a - d^2/b - m(\gamma - \alpha),$$

and corresponding generalized eigenvectors

$$\xi_1 = \left(b\,\frac{\tau_1}{\tau_2}, -d\,\frac{\tau_1}{\tau_2}, 1\right), \quad \xi_2 = (1, 0, -\frac{a_{31}}{a_{33}}), \quad \xi_3 = (0,0,1),$$

where $(a_{ij}) = (D\mathcal{X})^2$, $a_{33} = -b\lambda_3$,

$$\tau_1 = b^2(\alpha - \gamma)(ab - d^2 - bm(\gamma - \alpha)), \tag{4.4}$$

and

$$\begin{aligned} \tau_2 = 2(&-b^5\epsilon + (c-d)d^4\zeta + b^4(-d\delta + a\epsilon + 2d\zeta) \\ &+ b^2d^2(-d\delta - 2a\epsilon + c(\delta - 2\zeta) + d\zeta + a\eta) \\ &+ b^3d(k\alpha(\alpha-\gamma)^2 + a\delta + c\epsilon + d\epsilon - 2a\zeta - d\eta) \\ &+ bd^3(k\gamma(\alpha-\gamma)^2 - 2c\epsilon + 2d\epsilon + a\zeta + c\eta - d\eta)). \end{aligned} \tag{4.5}$$

Observe that $\lambda_3 = 0$ means that $m = (ab - d^2)/((\gamma - \alpha)b)$, which is the first coordinate of the point L_T. Consequently, our assumptions imply that $\lambda_3 \neq 0$, $a_{33} \neq 0$ and $\tau_1 \neq 0$. On the other hand τ_2 depends only on the parameters of the surface (2.2) and is non zero generically. On the singularity $(0,0,p_1)$ we have

$$\begin{aligned} \langle \nabla\mathcal{F}|_{(0,0,p_1)}, \xi_1\rangle &= \langle (\mathcal{F}_u, \mathcal{F}_v, \mathcal{F}_p), \left(b\frac{\tau_1}{\tau_2}, -d\,\frac{\tau_1}{\tau_2}, 1\right)\rangle, \\ &= \left(b\frac{\tau_1}{\tau_2}\right)(\mathcal{F}_u + p_1\mathcal{F}_v) + \mathcal{F}_p = 0. \end{aligned}$$

Therefore ξ_1 is tangent to the surface $S(\mathcal{I},\nu) = \mathcal{F}^{-1}(0)$ at $(0,0,p_1)$. ξ_3 is also tangent to $S(\mathcal{I},\nu)$ at the same point, because

$$\langle \nabla\mathcal{F}\,|_{(0,0,p_1)}, \xi_3\rangle = \langle(\mathcal{F}_u, \mathcal{F}_v, \mathcal{F}_p), (0,0,1)\rangle = \mathcal{F}_p = 0.$$

Consequently, the two eigenvectors $\xi_1,\ \xi_3$ generate the tangent plane to $S(\mathcal{I},\nu)$ at $(0,0,p_1)$. Now, to obtain an expression of $\mathcal{X}$ in terms of a base of this tangent plane we observe that

$$\mathcal{F}_v(0,0,p_1) = -a + (d^2/b) + m(\gamma - \alpha) = -\lambda_3 \neq 0.$$

It follows that $\mathcal{S}(\mathcal{I},\nu)$ is regular in a neighbourhood of $(0,0,p_1)$ and, by the implicit function theorem, there exists a differentiable function $v = v(u,p)$ in a neighborhood of $(0,0,p_1)$ such that $\mathcal{F}(u, v(u,p), p) = 0$. Then a local analysis in the chart (u,p) of the singularity $(0,0,p_1)$ of $\mathcal{X}$ over the surface $S(\mathcal{I},\nu)$ will be sufficient. Straightforward computations show that the Taylor expansion of $v(u,p)$ is of the form

$$\begin{aligned} v(u,p) = \frac{1}{b\,(b^2 - cd + dn(\alpha - \gamma))}&(-(b^2d + d^3)u \\ &+ (ab^2 - b^3 - 2bd^2 + b^2m\alpha - b^2m\gamma)up \\ &+ (bdk^3 + bdk\alpha^2 + bdm(\zeta - \delta) + m\epsilon(d^2 - b^2))u^2) + \mathcal{O}(3), \end{aligned}$$

and that the restriction of the vector field $\mathcal{X}$ to the implicit surface $v = v(u,p)$ has linear part with eigenvalues

$$\bar{\lambda}_1 = 0, \quad \bar{\lambda}_3 = a - d^2/b - m(\gamma - \alpha) \neq 0,$$

and corresponding eigenvectors

$$\bar{\xi}_1 = (b\,\tau_1/\tau_2, 1) \quad \bar{\xi}_3 = (0,1).$$

Consequently, all center manifolds W^c are tangent to the line

$$p = \left(\frac{\tau_2}{\tau_1 b}\right) u.$$

By invariant manifold theory (see, for example [7], chapter 3) $\mathcal{X} \mid W^c$ is topologically equivalent to

$$\dot{u} = a_2 u^2 + \mathcal{O}\left(2\right)$$

near the singularity. On the other hand, the generic condition $\tau_1\tau_2 \neq 0$ guarantees that the linear change of coordinates $(u, p) \longmapsto (x_1, x_2)$ represented by the matrix

$$\begin{pmatrix} b\tau_1 & -\tau_2 \\ \tau_2 & b\tau_1 \end{pmatrix}$$

gives a topologically equivalent vector field X in the chart (x_1, x_2) such that its linear part has eigenvectors coincident with the canonical orthonormal basis and its components X_1, X_2 in that basis satisfies

$$\frac{\partial X_1}{\partial x_1}(0,0) = \frac{\partial X_1}{\partial x_2}(0,0) = 0,\ \frac{\partial X_2}{\partial x_2}(0,0) \neq 0,\ \text{and}\ \frac{\partial^2 X_1}{\partial x_1^2}(0,0) \neq 0.$$

Therefore, the vector field X have a saddle-node singular point at $(0,0)$, according to [19], section I.3. Then $(0, 0, p_1)$ is a saddle-node singular point of $\mathcal{X}$. For the roots p_2 and p_3 the non zero eigenvalues of $\mathcal{X}$ are

$$\begin{aligned}\lambda_i = \pm\frac{1}{2d^2}&(a^2b - 2ab^2 + b^3 + 4bd^2 + 2bm\alpha(a-b)\\ &+ bm^2(\alpha^2+\gamma^2) + 2bm\gamma(-a+b+m\alpha)\\ &+ \sqrt{4d^2 + (a-b+m(\alpha-\gamma))^2}(ab - b^2 - 2d^2 + bm(\alpha-\gamma))),\end{aligned}$$

for $i = 2, 3$. It follows that $(0, 0, p_i)$ are saddle singularities for $i = 2, 3$. And the surface $\mathcal{F}(u, v, p) = 0$ is not regular at these points, because

$$\nabla\mathcal{F}(0,0,p) = (dp^2 + p(a-b+m(\alpha-\gamma)) - d, \frac{b}{d}(dp^2 + p(a-b+m(\alpha-\gamma)) - d), 0),$$

which is $(0, 0, 0)$ for $p = p_i,\ i = 2, 3$. Straightforward computation shows that the Hessian determinant

$$\Delta = Det(Hess(\mathcal{F})(0,0,p_i)) \tag{4.6}$$

of the function $\mathcal{F}$ at these points is given by

$$\begin{aligned}\Delta = \frac{1}{d^2(\alpha-\gamma)}&(2(a-b+2dp_i+m(\alpha-\gamma))^2\\ &(b^3(\epsilon - p_i^2\epsilon + p_i(\zeta-\delta)) + d^2(k^3p_i(\gamma-\alpha)\\ &+ kp_i\gamma^2(\gamma-\alpha) - (c-d)(p_i^2\zeta + p_i(\epsilon-\eta) - \zeta))\\ &+ bd(-k^3(p_i^2-1)(\alpha-\gamma) - k(p_i^2-1)\alpha(\alpha-\gamma)\gamma\\ &- dp_i\delta + d\epsilon - ap_i\epsilon - dp_i^2\epsilon + a\zeta + dp_i\zeta - ap_i^2\zeta\\ &+ c(p_i\delta - \epsilon + p_i^2\epsilon - p_i\zeta) + ap_i\eta) + b^2(k^3p_i(\alpha-\gamma)\\ &+ kp_i\alpha^2(\alpha-\gamma) + a(p_i\delta - \epsilon + p_i^2\epsilon - p_i\zeta)\\ &+ d(p_i\epsilon - \zeta + p_i^2\zeta - p_i\eta)))),\end{aligned}$$

which we are assuming different from zero. Consequently, the points $(0, 0, p_i),\ i = 2, 3$ are non degenerate critical points of $\mathcal{F}$ of Morse type and index 1 or 2, since $S(\mathcal{I}, \nu) = \mathcal{F}^{-1}(0)$ contains the projective line (see, for example [13]). Then we have in our setting the same geometric and dynamical properties obtained in [12] for surfaces in $\mathbb{R}^3$ which characterize the principal configuration showed in Figure 3. ■

Observe that Δ contains all the parameters of the surface and also depends on the parameter m of the normal vector field. Therefore $\Delta \neq 0$ except at one hypersurface in the space of all parameters. Then we have that the hypotheses of Theorem 4.2 are generically satisfied.

Now, we are going to apply one result of [19] to determine the bifurcation that occurs at the points of $L - L_T$ in the space $\mathcal{N}_1$.

First observe that, from the 2-jet of (2.5) the curves

$$B(u, v(u)) = 0,$$

and

$$n(u) = C(u, v(u)),$$

where

$$v(u) = \frac{du(b - a + m(\gamma - \alpha) + ku(k^3 + \alpha^2) + mu(\zeta - \delta))}{b(a - b + m(\alpha - \gamma))} + \mathcal{O}(2) \tag{4.7}$$

have a quadratic contact at the origin, because

$$n(u) = -u^2 \left(m\epsilon + d \left(\frac{k^3 + k(\alpha^2 + \beta^2) + m(\zeta - \delta)}{a - b + m(\alpha - \gamma)} \right) \right). \tag{4.8}$$

Then, we have the following.

Theorem 4.3. *Let $I^r(M)$ be the set of all immersions of class C^r of M in $\mathbb{R}^4$ and $\mathcal{I} \in I^r(M)$, with $r \geq 5$, such that $\mathcal{I}$ satisfy the condition $\mathbf{D_{23}}$ at a ν-umbilical point x for $\nu \in \mathcal{N}_1$. Then there exists a real valued function $\mathcal{B}$ of class C^{r-2} in a neighborhood $\mathcal{V}$ of $\mathcal{I}$ and a neighborhood $\mathcal{W}$ of x such that*

I) $d\mathcal{B} \neq 0$.

II) $\mathcal{B}(\beta) > 0$ if and only if $\beta \in I^r(M)$ does not have ν-umbilical points on $\mathcal{W}$.

III) $\mathcal{B}(\beta) < 0$ if and only if $\beta \in I^r(M)$ has two ν-umbilical points of type D_2 and D_3.

IV) $\mathcal{B}(\beta) = 0$ if and only if $\beta \in I^r(M)$ has only one ν-umbilical point at $\mathcal{W}$ and it is of type D_{23}.

Proof. Let

$$f_\lambda(u, v) = (u, v, \varphi(u, v) + \lambda uv, \psi(u, v)) \tag{4.9}$$

be a one-parameter family of immersions β in neighborhoods $\mathcal{V}$ of $\mathcal{I}$ and $\mathcal{W}$ of $x \in M$, respectively. Then, since the coefficients A, B and C of the differential equation (2.5) for this family becomes zero at a ν-umbilical point, the ν-umbilical points of β are defined by

$$B_\lambda(u, v) = 0,$$
$$C_\lambda(u, v) = 0.$$

By (4.7) and (4.8) we have in a neighborhood of $\beta(x)$ that $B_\lambda(u, v_\lambda(u)) = 0$, with

$$v_\lambda(u) = \frac{du(b - a + m(\gamma - \alpha) + ku(k^3 + \alpha^2) + k\lambda^2 + mu(\zeta - \delta))}{b(a - b + m(\alpha - \gamma))} + \mathcal{O}(2).$$

Finally, if $\mathcal{B}(\beta) = n_\lambda(u_\lambda)$, where u_λ is the unique critical point of $C_\lambda(u, v_\lambda)$, a direct calculation shows that

$$\frac{d\mathcal{B}(\beta)}{d\lambda} = -1 \neq 0,$$

which imply the result by Lemma 3.2 of [19]. ■

Therefore, under our generic assumptions, the non simple ν-umbilical points different from L_T are codimension one bifurcation points in the space $\mathcal{N}_1$ and have ν-principal configuration of type D_{23}.

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