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Imaginary quadratic function fields with ideal class group of prime exponent

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ABSTRACT

For K, an imaginary quadratic extension of a rational function field over a finite field, in which the infinite place ramifies, we give necessary conditions (illustrating for exponent three) for the ideal class group to have odd prime exponent. For exponent two we classify all such extensions, taking this opportunity to complete the list that we previously had.

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1. Introduction

Consider a rational function field of one variable F over its field of constants $\mathbb{F}_q$, where $\mathbb{F}_q$ is a finite field of $q = p^n$ elements, p a prime number, $n \geq 1$. Once we choose a generator x of F over $\mathbb{F}_q$, it is customary to call those places of F coming from irreducible primes of $A = \mathbb{F}_q[x]$ the finite places, and the unique remaining place is called the infinite place and is denoted by $\mathfrak{p}_\infty$ or $\mathfrak{p}_{1/x}$ because it has uniformizer $1/x$.

Artin [1] considered the analogy $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R} \leftrightarrow A = \mathbb{F}_p[x]$, $F = \mathbb{F}_p(x)$, $F_\infty = \mathbb{F}_p((1/x))$, p an odd prime, and called a quadratic extension K of F *real* if $K \subset F_\infty$, i.e., there are two places of K over $\mathfrak{p}_\infty$ (i.e., $\mathfrak{p}_\infty$ splits completely) and *imaginary* if there is only one place $\mathfrak{P}_\infty$ above $\mathfrak{p}_\infty$. MacRae [6] calls K imaginary if further $\mathfrak{p}_\infty$ ramifies in K. In this article, we follow Artin's definition.

If K is not a geometric quadratic extension of $\mathbb{F}_q(x)$, then K is $\mathbb{F}_{q^2}(x)$, which is understood, so we restrict to geometric imaginary quadratic extensions only.

We can consider the divisor class group $C_0(K)$ of divisors of degree 0 of K or the ideal class group $C(\mathcal{O}_K)$ of the integral closure $\mathcal{O}_K$ of $\mathbb{F}_q[x]$ in K. We know these are finite and $|C(\mathcal{O}_K)| = (\deg \mathfrak{P}_\infty)\,|C_0(K)|$. Thus, when we consider ideal class groups of exponent ℓ, ℓ an odd prime, we automatically reduce to the ramified case.

Often, e.g. in Drinfeld's theory, any chosen place is called the infinite place and others are called finite places, but we stick to this classical terminology, which practically means that any place of degree one can be chosen to be the infinite place, given just the abstract rational function field. Hence K is called [8] a *totally imaginary* extension of F if no place of degree one in F splits in K.

To classify relative extensions K/F, one uses isomorphisms keeping F (and hence $\mathbb{F}_q$) constant, but to shorten the list or to use abstract K's one often uses the notion of isomorphism instead. Here, we classify function fields up to isomorphism.

2. Norms of integral elements in imaginary extensions

Let K be an imaginary extension of $\mathbb{F}_q(x)$ with genus g. There is only one place $\mathfrak{P}_\infty$ which lies over the infinity place $\mathfrak{p}_\infty$ of $\mathbb{F}_q(x)$. The following theorem, provides a bound on the degree of the norm of integral elements.

Theorem 2.1. *[8, Theorem 4, p. 220] If K is an imaginary quadratic extension of $\mathbb{F}_q(x)$, then, for any $\alpha \in K \setminus \mathbb{F}_q[x]$, which is integral over $\mathbb{F}_q[x]$,*

$$\deg N(\alpha) \geq 2g + 1,$$

where N is the norm of α.

Corollary 2.2. *If K is an imaginary quadratic extension of $\mathbb{F}_q(x)$ for which the divisor class group has exponent e, then, for any finite place $\mathfrak{p}$ of $\mathbb{F}_q(x)$ which splits in K,*

$$\deg_{\mathbb{F}_q(x)} \mathfrak{p} > \frac{2g}{ed_\infty}$$

where g is the genus of K and $d_\infty = \deg \mathfrak{P}_\infty$ is the degree of the place of K over the infinite place $\mathfrak{p}_\infty$ of $\mathbb{F}_q(x)$.

Proof. Let $p(x)$ be the irreducible polynomial of $\mathbb{F}_q[x]$ associated with the divisor $\mathfrak{p}$, and let $\mathfrak{P}$ be any place of K which lies over $\mathfrak{p}$. Thus,

$$\mathfrak{P}^{d_\infty}/\mathfrak{P}_\infty^{\deg \mathfrak{p}} \in C_0(K).$$

Now, since the exponent of the divisor class group is e,

$$\left(\mathfrak{P}^{d_\infty}/\mathfrak{P}_\infty^{\deg \mathfrak{P}}\right)^e = (\alpha), \quad \alpha \in K.$$

Taking norms of both sides gives

$$\left(\mathfrak{p}^{d_\infty}/\mathfrak{p}_\infty^{d_\infty \deg \mathfrak{p}}\right)^e = (N(\alpha)).$$

But then,

$$N(\alpha) = a \cdot p(x)^{ed_\infty}$$

for some $a \in \mathbb{F}_q$. By Theorem 2.1,

$$ed_\infty \deg p(x) \geq 2g + 1.$$

Thus

$$\deg \mathfrak{p} > \frac{2g}{ed_\infty}. \quad \square$$

Theorem 2.3. *Let K be a quadratic extension of F for which the divisor class group has exponent e, then K is a totally imaginary extension of F if K has genus greater than or equal to e.*

Proof. By hypothesis, K is a quadratic extension of F for which the divisor class group has exponent e. Suppose there is a place of degree 1 in F which splits in K; let it be the infinite place of F. Then, if $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are two places of K which lie over the infinite place of F,

$$\mathfrak{Q}_1/\mathfrak{Q}_2 \in C_0(K).$$

Since the divisor class group has exponent e, it follows that

$$\mathfrak{O}_1^e/\mathfrak{O}_2^e = (\alpha).$$

Hence, $v_{\mathfrak{O}_2}(\alpha) = -e$. If the characteristic is odd, by [7, Theorem 4] we have

$$e \geq 1 + g > g$$

and if the characteristic is 2, from [7, Theorem 10] follows that

$$e \geq g + \frac{1}{2} > g.$$

Thus, if a place of degree 1 splits in such a K, K has genus less than e. □

Theorem 2.4. *Let K be a quadratic extension of F then, for a proper choice of the generator x of F over $\mathbb{F}_q$, K is an imaginary extension of $\mathbb{F}_q(x)$ if $q > 2g^2 + 2g\sqrt{g^2-1} - 1$.*

Proof. Suppose that all the places of degree 1 in F split in K. Then K has $2q+2$ places of degree one. However, by the Riemann hypothesis for function fields,

$$|(2q+2) - (q+1)| \leq 2gq^{1/2}$$
$$|q+1| \leq 2gq^{1/2}.$$

Hence,

$$q + 1 - 2gq^{1/2} \leq 0.$$

Now, let $y = \sqrt{q}$. Then $y^2 - 2gy + 1 \leq 0$, and the roots of this equation are $y = g \pm \sqrt{g^2-1}$. So $q = 2g^2 \pm 2g\sqrt{g^2-1} - 1$. But, if $q > 2g^2 + 2g\sqrt{g^2-1} - 1$, then $q + 1 - 2gq^{1/2} > 0$, a contradiction. Therefore, some place of degree one in F must not split in K. □

From algebraic geometry [10], it is known that the ℓ-rank of the divisor class group of an algebraic function field over an algebraically closed field of constants is $2g$ if ℓ is not the characteristic, and it is at most g when ℓ is the characteristic. Thus, in a function field over a finite field the ℓ-rank of the divisor class group is at most $2g$ when ℓ is not the characteristic and at most g when ℓ is the characteristic.

Theorem 2.5. *If $K/\mathbb{F}_q$ is an algebraic function field over the finite field $\mathbb{F}_q$, $q = p^n$, in which the divisor class group has exponent ℓ, with ℓ prime, then,*

$$\begin{cases} q = \ell & \text{if } p = \ell \text{ and } \ell > 2 \\ q = 2 \text{ or } 4 & \text{if } p = \ell \text{ and } \ell = 2 \\ q \leq (\ell+1)^2 & \text{if } p \neq \ell. \end{cases}$$

Proof. If $p \neq \ell$, then for $h_K = |C_0(K)|$, the class number of K,

$$h_K \leq \ell^{2g}$$

since $C_0(K)$ is at most the product of $2g$ copies of $\mathbb{Z}/\ell\mathbb{Z}$. However, by the Riemann hypothesis for function fields,

$$(q^{\frac{1}{2}} - 1)^{2g} \leq h_K \leq \ell^{2g}. \tag{1}$$

This implies $\log_\ell(\sqrt{q} - 1) \leq 1$, so

$$q \leq (\ell + 1)^2.$$

If $p = \ell$, then

$$(q^{\frac{1}{2}} - 1)^{2g} \leq h_K \leq \ell^g$$

or, equivalently

$$q \leq (\sqrt{\ell} + 1)^2. \tag{2}$$

But, since $q = p^n$ for some positive integer n, we have from (2) that

$$p^n \leq (\sqrt{p} + 1)^2.$$

Now, $(\sqrt{p}+1)^2 < p^2$ if $p > 2$, so in this case $n = 1$ and $q = p$. When $p = 2$, $(\sqrt{2}+1)^2 < 2^3$ and therefore $q = 2$ or $q = 4$. □

This theorem says that there is only a finite number of q's. Next, we show that for each order there is only a finite number of possible K, by bounding the genus of such fields.

If K is such that not all the places of degree 1 of $\mathbb{F}_q(x)$ are inert in K and such that the genus $g \geq e$, then, by Theorem 2.3, some place of degree 1 must ramify. Thus it can be assumed that the field K is imaginary and that the infinite place of $\mathbb{F}_q(x)$ ramifies in K. In this case, Corollary 2.2 says that no place of degree $\frac{2}{e}g$ or less can split in K.

On the other hand, Theorem 6 from [7] says that for a fixed q, if m is a positive integer such that

$$q^m - 2gq^{\frac{m}{2}} - 2m(g + 2) \geq 0$$

then there exists a place in $\mathbb{F}_q(x)$ which splits completely in K and which has degree less than m_0, where $m_0 = m + 2$. Now, if we take $m = \left\lfloor \frac{2g}{e} \right\rfloor - 1$, where $\lfloor r \rfloor$ denotes the floor of the rational number r, we have

$$m+1=\left\lfloor \frac{2g}{e} \right\rfloor \leq \frac{2}{e}g.$$

Moreover, if $g \geq e$ then $m > 0$. This implies

Theorem 2.6. *Let K be a quadratic extension of $\mathbb{F}_q(x)$ of genus g, in which not all the places of degree 1 in $\mathbb{F}_q(x)$ are inert. If the divisor class group of K has exponent e, and*

$$q^m - 2gq^{\frac{m}{2}} - 2m(g+2) \geq 0 \tag{3}$$

where $m = \left\lfloor \frac{2g}{e} \right\rfloor - 1$, then exists a place in $\mathbb{F}_q(x)$ which splits completely in K and which has degree less or equal than $\frac{2}{e}g$ whenever $g \geq e$.

3. Characterization of quadratic extensions of prime exponent

Theorem 3.1. *Let K be a quadratic extension of $\mathbb{F}_q(x)$, in which the infinite place ramifies, and let $h = |C(\mathcal{O}_K)|$ be its class number. Let t be the number of places of $\mathbb{F}_q(x)$ that ramify in K. Then $h = h_K$ and*

(a) *The group of divisor classes of degree zero has exponent 2 if and only if $h = 2^{t-2}$ or $h = 2^{t-1}$, depending on whether $q \equiv 1 \bmod 2$ or not.*
(b) *If the group of divisor classes of degree zero has exponent an odd prime number, then $t = 2$ or 1, depending on whether $q \equiv 1 \bmod 2$ or not.*

Proof. The first claim follows from $h = d_\infty \cdot h_K$ since in this case, when the infinite place ramifies, $d_\infty = 1$. For (a) see Theorem 2 in [2]. The part (b) follows from Theorem 9 in [12]. □

The converse of the Theorem 3.1(b) is not true, as shown in the following examples:

Example 3.2.

(a) Let $q = 3$. Then the function field $\mathbb{F}_3(x, y)$, where $y^2 + 2x^5 + 2x^3 + 2x^2 + 1 = 0$, is a quadratic extension of $\mathbb{F}_3(x)$ of genus 2, where two places of $\mathbb{F}_3(x)$ ramify, but its exponent is 9, not a prime.
(b) Now, let $q = 2$. Then the function field $\mathbb{F}_2(x, y)$, where $y^2+y+x^{13}+x^7+x^6+x^3+1 = 0$, is a quadratic extension of $\mathbb{F}_2(x)$ of genus 6, where one place of $\mathbb{F}_2(x)$ ramifies, but its exponent is 49, not a prime.

4. Exponent three

Observe that Theorems 2.5 and 2.6 give an algorithm to find bounds on q and, in each case, bounds for the genus of K. Next, we apply this to the particular case when the exponent equals three.

Theorem 4.1. *If K is an algebraic function field over the finite field $\mathbb{F}_q$ in which the divisor class group has exponent 3, then*

$$q \in \{2, 3, 4, 5, 7, 8, 11, 13, 16\}.$$

Proof. From Theorem 2.5, we immediately have $q = 3$ if $p = 3$ and $q \leq 16$ otherwise. □

Now, we have

Theorem 4.2. *If K is a quadratic imaginary extension of $\mathbb{F}_q(x)$ of genus g, in which the infinite place of $\mathbb{F}_q(x)$ ramifies, and if the ideal class group of K has exponent 3, then:*

(1) $q = 16$, $1 \leq g \leq 2$
(2) $q = 13$, $1 \leq g \leq 2$
(3) $q = 7$, $1 \leq g \leq 4$
(4) $q = 5$, $1 \leq g \leq 7$
(5) $q = 4$, $1 \leq g \leq 8$
(6) $q = 3$, $1 \leq g \leq 11$
(7) $q = 2$, $1 \leq g \leq 19$.

Proof. Firstly, we prove the theorem for $q = 16$. In this case, by Equation (1), $h_K = 3^{2g}$. Let $L(t) = 1 + a_1 t + \cdots + a_{2g} t^{2g} \in \mathbb{Z}[t]$ be the L-polynomial of K. This polynomial factors in $\mathbb{C}[t]$ in the form

$$L(t) = \prod_{j=1}^{2g} (1 - \alpha_j t).$$

Now, by the Hasse–Weil theorem, the reciprocals of the roots of $L(t)$ satisfy

$$|\alpha_j| = q^{1/2} = 4, \ j = 1, 2, \ldots, 2g,$$

and so

$$\alpha_j = 4e^{i\theta_j}, \ j = 1, 2, \ldots, 2g.$$

Since $L(1) = h_K = \prod_{j=1}^{2g} (1 - 4e^{i\theta_j})$, arranging by conjugate pairs, we get that

$$9^g = \prod_{j=1}^{g} (17 - 8\cos(\theta_j)). \tag{4}$$

But $17 - 8\cos(\theta_j) \geq 9$ and therefore, by (4), $17 - 8\cos(\theta_j) = 9$, i.e., $\cos(\theta_j) = 1$, $j = 1, 2, \ldots, 2g$. This shows that $\alpha_j = 4$, $j = 1, 2, \ldots, 2g$ and $L(t) = (1-4t)^{2g} = 1-8gt+\cdots$. Since N_1, the number of places of degree one of K, is $N_1 = a_1 + q + 1 = -8g + 17 \geq 0$, it follows that $g \leq 2$.

Now, we consider the case $q = 13$. Since the Equation (3) holds for $g \geq 5$, Theorem 2.6 guarantees the existence of a place $\mathfrak{p}$ in $\mathbb{F}_q(x)$ of degree $\leq 2g/3$ which splits in K. On the other hand, Corollary 2.2 establishes that $\mathfrak{p}$ must have degree $> 2g/3$ which is a contradiction. Therefore, if K has exponent 3, then $1 \leq g \leq 19$.

The same argument shows that if $q = 2, 3, 4, 5, 7$, then the genus is less than or equal to $19, 11, 8, 7, 5$, respectively.

Next, assume K is an imaginary quadratic extension of $\mathbb{F}_{13}(x)$ of genus $g = 3$. Then, $K = \mathbb{F}_{13}(x, y)$ with $y^2 = f(x)$, where $f(x)$ is an irreducible polynomial of degree 7 over $\mathbb{F}_{13}$. By Theorem 3.1, $f(x)$ and the infinite place are the only places that ramify in K. According to Corollary 2.2, the finite places in $\mathbb{F}_{13}(x)$ of degree 1 and 2 are inert in K. So, K has exactly one rational place and 13 places of degree 2. Since $\mathcal{N}_r$, the number of places of degree one in the constant field extension $K\mathbb{F}_{13^r}$, is $\sum_{d|r} d \cdot N_r$ where N_r is the number of places of degree r in K, it follows that $\mathcal{N}_2 = 27$. But by the Hasse–Weil bound for $K\mathbb{F}_{13^2}$, $\mathcal{N}_2$ satisfies the inequality $\left|\mathcal{N}_2 - (13^2 + 1)\right| \leq 2 \cdot 13g$, which gives a contradiction. A very similar argument excludes the case $g = 4$.

For the values of $q = 8, 11$, mentioned in Theorem 4.1, there are no quadratic function fields at all with class group of exponent 3. This follows because, according to [5, Theorem 3.2], if 3 divides $q + 1$, the 3-rank of the class group doubles when going from K to the quadratic constant field extension $L = K\mathbb{F}_{q^2}$. Since the genus g of K and L is the same, and for L the 3-rank is bounded by $2g$, the 3-rank for K is bounded by g. So from the inequality $(\sqrt{q} - 1)^{2g} \leq 3^g$, which is not valid for $q = 8, 11$, the claim is proved.

Next, assume $q = 7$ and $g = 5$. Then, from Table 3 of [13], the 3-rank is bounded by 4. Therefore, a class group of exponent 3 for $q = 7$, $g = 5$ would lead to the contradiction $(\sqrt{7} - 1)^{10} \leq 3^4$. □

Among all the possibilities given in Theorem 4.2, we completely examine, using the computer algebra package Magma [4] and the mathematics software system SageMath [14], the following cases:

q	genus
2	$1 \leq g \leq 6$
3	$1 \leq g \leq 4$
4	$1 \leq g \leq 2$
5	$1 \leq g \leq 4$
7	$1 \leq g \leq 3$
13	$1 \leq g \leq 2$
16	$1 \leq g \leq 2$

Next, we list the extensions found with exponent three, up to isomorphism.

q	g	h	Extension
2	1	3	$y^2 + y + x^3 = 0$
	2	3	$y^2 + y + x^5 + x + 1 = 0$
	3	3	$y^2 + y + x^7 + x^3 + 1 = 0$
	4, 5, 6		Computer search shows that there are no such extensions
3	1	3	$y^2 + 2x^3 + 2x^2 + 1 = 0$
	2	3	$y^2 + 2x^5 + 2x^3 + 2x + 1 = 0$
	3, 4		Computer search shows that there are no such extensions
4	1	3	$y^2 + y + wx^3 = 0$, where $w^2 + w + 1 = 0$
	1	9	$y^2 + y + x^3 = 0$
	2	9	$y^2 + y + wx^5 + x^3 + w^2x + w = 0$, where $w^2 + w + 1 = 0$
5	1	3	$y^2 + 4x^3 + x + 3 = 0$
	2, 3, 4		Computer search shows that there are no such extensions
7	1	3	$y^2 + 6x^3 + 3 = 0$
	1	9	$y^2 + 6x^3 + 5 = 0$
	2, 3		Computer search shows that there are no such extensions
13	1	9	$y^2 + 12x^3 + 10 = 0$
	2		Computer search shows that there are no such extensions
16	1	9	$y^2 + y + w^3x^3 = 0$, where $w^4 + w + 1 = 0$
	2	81	$y^2 + y + w^5x^5 + w^3 = 0$, where $w^4 + w + 1 = 0$

Remark 4.3. By Theorem 4.2, it follows that the examples listed in the above table are the only ones for $q > 7$.

Theorem 4.4. *Up to isomorphism, there are precisely eight imaginary quadratic function fields of class number three, in which the infinite place ramifies. The complete list is given in the following table*

Imaginary quadratic function fields with class number 3.

q	g	Extension
2	1	$y^2 + y + x^3 = 0$
	2	$y^2 + y + x^5 + x + 1 = 0$
	3	$y^2 + y + x^7 + x^3 + 1 = 0$
3	1	$y^2 + 2x^3 + 2x^2 + 1 = 0$
	2	$y^2 + 2x^5 + 2x^3 + 2x + 1 = 0$
4	1	$y^2 + y + wx^3 = 0$, where $w^2 + w + 1 = 0$
5	1	$y^2 + 4x^3 + x + 3 = 0$
7	1	$y^2 + 6x^3 + 3 = 0$

Proof. According to [11] and [3], there are thirteen quadratic function fields with class number three, up to isomorphism. It is easily checked that, up to isomorphism, there are exactly eight imaginary quadratic function fields of exponent three in which the infinite place ramifies. The above table shows a list of representatives of the distinct isomorphism classes. □

Remark 4.5 *([11, Theorem 3.2, p. 641])*. Up to $\mathbb{F}_q$-isomorphism, there is an extra imaginary quadratic function field with class number 3 and $g = 1$ and is given by $\mathbb{F}_4(x, y)$ where $y^2 + y + w^2x^3 = 0$ and $w^2 + w + 1 = 0$.

5. Exponent two

We take this opportunity to thank Andreas Schweizer for having pointed out the incompleteness of the list we had given about the quadratic function fields whose ideal class group have exponent 2 given in [2]. See his article [9, p. 1019]. For this reason we check the results obtained in [2, Theorem 21], and we realized that the cases $h = 16$, $q = 2$, $g = 9$ and $h = 32$, $q = 3$, $g = 5$ are missing. Then, we re-do the computer calculations systematically and found that indeed, we omitted some examples when we tried for the first time.

We give now, up to isomorphism (indeed up to $\mathbb{F}_q$-isomorphism), the complete list of 19 quadratic function fields whose ideal class group has exponent two and the infinite place of $\mathbb{F}_q(x)$ ramifies:

Class number 2.

q	g	Extension
2	1	$y^2 + xy + x(x^2 + x + 1) = 0$
	2	$y^2 + xy + x(x^4 + x + 1) = 0$
	2	$y^2 + (x^2 + x + 1)y + (x^2 + x + 1)(x^3 + x + 1) = 0$
	3	$y^2 + (x^2 + x + 1)y + (x^2 + x + 1)(x^5 + x^2 + 1) = 0$
	3	$y^2 + (x^3 + x^2 + 1)y + (x^3 + x^2 + 1)(x^4 + x^3 + 1) = 0$
3	1	$y^2 - (x + 2)(x^2 + 1) = 0$
4	1	$y^2 + xy + wx(x^2 + wx + w) = 0$, where $w^2 + w + 1 = 0$
5	1	$y^2 - x(x^2 + 2) = 0$

Class number 4.

q	g	Extension
2	2	$y^2 + x(x + 1)y + x(x + 1)(x^3 + x + 1) = 0$
	3	$y^2 + x(x + 1)y + x(x + 1)(x^5 + x^3 + x^2 + x + 1) = 0$
	3	$y^2 + x(x^2 + x + 1)y + x(x^2 + x + 1)(x^4 + x + 1) = 0$
3	1	$y^2 - x^3 + x = 0$
	2	$y^2 - 2x(x + 1)(x^3 + x^2 + x + 2) = 0$
	2	$y^2 - 2(x + 2)(x^2 + 1)(x^2 + x + 2) = 0$
5	1	$y^2 - x^3 - x = 0$
7	1	$y^2 - x^3 + 1 = 0$
9	1	$y^2 - x^3 + \sqrt{-1}\,x = 0$

Class number 8.

q	g	Extension
3	2	$y^2 - x(x + 1)(x + 2)(x^2 + 1) = 0$

Class number 16.

q	g	Extension
5	2	$y^2 + 4x^5 + x = 0$

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