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Duality of the Kulkarni Limit Set for Subgroups of PSL(3, $\mathbb{C}$)

Waldemar Barrera[1] · **Adriana Gonzalez-Urquiza**[2] · **Juan Pablo Navarrete**[1]

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Abstract In this paper we give a generalization of the Conze–Guivarc'h limit set. With this definition the limit set has very similar properties to those of the limit set in hyperbolic spaces. Moreover, we prove a relation between this new limit set and the Kulkarni limit set. Additionally we show that some closed subsets can be approximated by the Conze–Guivarc'h limit set. This is a result in the theory of classic Kleinian groups.

Keywords Kleinian groups · Dual projective complex plane · Limit set

1 Introduction

The discrete subgroups of PSL$(2, \mathbb{C})$ acting on the sphere $\mathbb{S}^2$ with non-empty region of discontinuity are known as *Kleinian groups*. One of the objects to study within this action is the limit set. A point x is a *limit point* for the Kleinian group G if there is a point $z \in \mathbb{S}^2$ and a sequence (g_m) of distinct elements of G with $g_m(z) \to x$. The set of limit points is called the *limit set*, denoted by $\Lambda(G)$.

On the other hand, complex Kleinian groups were introduced by Seade and Verjovsky as discrete subgroups G of PGL$(n+1, \mathbb{C})$ acting properly discontinuously on some non-empty G-invariant open subset of the n-th dimensional complexprojective

✉ Adriana Gonzalez-Urquiza
adriana.gurquiza@im.unam.mx

Waldemar Barrera
bvargas@correo.uady.mx

Juan Pablo Navarrete
jp.navarrete@correo.uady.mx

1 Facultad de Matemáticas, UADY, Mérida, Mexico

2 Instituto de Matemáticas, Unidad Cuernavaca, UNAM, Cuernavaca, Mexico

space $\mathbf{P}^n_{\mathbb{C}}$. Since the beginning of the study of these subgroups, the limit set that has been considered is the one defined by Kulkarni (1978). Roughly speaking, he took into account not only the closure of accumulation points of orbits of elements in $\mathbf{P}^n_{\mathbb{C}}$ but also the accumulation points of orbits of compact sets. We denote this set by $\Lambda_K(G)$ and call it the *Kulkarni limit set*. See Sect. 2.5 for the definition.

It is shown in Barrera et al. (2011), that under some hypothesis, the Kulkarni limit set of a subgroup G of PSL$(3, \mathbb{C})$ is made up of projective lines, for example when G is a discrete infinite group without global fixed points nor invariant complex projective lines (see Barrera et al. 2011, Theorem 1.3). There are also another technical criteria to guarantee that the Kulkarni limit set is a union of complex projective lines. For example, Theorem 1.2 in Barrera et al. (2011) requires that the Kulkarni limit set and another set contain at least three complex projective lines.

A classical result in projective geometry is the natural identification of $\mathbf{P}^2_{\mathbb{C}}$ and the dual projective space $(\mathbf{P}^2_{\mathbb{C}})^*$. The dual space $(\mathbf{P}^2_{\mathbb{C}})^*$ can be considered as the space consisting of all complex projective lines in $\mathbf{P}^2_{\mathbb{C}}$ and the complex projective line ℓ, with equation $Ax + By + Cz = 0$, is identified with the point $[A : B : C] \in \mathbf{P}^2_{\mathbb{C}}$.

Under this identification, the natural action of $g \in$ PSL$(3, \mathbb{C})$ on the space of complex projective lines $(\mathbf{P}^2_{\mathbb{C}})^*$ is given by

$$g \cdot \ell \longleftrightarrow (\mathbf{g}^{-1})^T \begin{pmatrix} A \\ B \\ C \end{pmatrix},$$

where $\mathbf{g}$ denotes a matrix in SL$(3, \mathbb{C})$ which induces the projective transformation g.

Finally, when F is a subset of $(\mathbf{P}^2_{\mathbb{C}})^*$ we denote, by abuse of notation,

$$\bigcup_{\ell \in F} \ell$$

the subset of $\mathbf{P}^2_{\mathbb{C}}$ obtained as the union of all the lines (considered as subsets of $\mathbf{P}^2_{\mathbb{C}}$) determined by the elements in F.

In Conze and Guivarc'h (2000) the authors give a definition of limit set for the action of a group on a linear space. They consider the closure of attracting fixed points of proximal elements in the group (an element is called proximal whenever it has an eigenvalue with modulus strictly greater than the modulus of all other eigenvalues). Every proximal element is a loxodromic element according to the classification given in Navarrete (2008), but the converse is not true. On the other hand, not every subgroup of PSL$(3, \mathbb{C})$ contains proximal elements, so we define the limit set $\hat{L}(G)$ of a group G acting on $(\mathbf{P}^2_{\mathbb{C}})^*$ (see Definition 6), even if G does not contain proximal elements. This limit set $\hat{L}(G)$ has similar properties to those of the limit set of the subgroups of PSL$(2, \mathbb{C})$, and we present them in Corollaries 1, 2 and 3.

In the following theorem, we show a relation between the limit set $\hat{L}(G)$ of G acting on $(\mathbf{P}^2_{\mathbb{C}})^*$ and the Kulkarni limit set $\Lambda_K(G)$ of G acting on $\mathbf{P}^2_{\mathbb{C}}$.

Theorem 1 *Let $G \leq PSL(3, \mathbb{C})$ be an infinite discrete subgroup acting on $\mathbf{P}^2_{\mathbb{C}}$ without fixed points nor invariant lines. Let $\hat{L}(G)$ be the limit set (according to Definition 6) of G acting on $(\mathbf{P}^2_{\mathbb{C}})^*$, then*

$$\Lambda_K(G) = \bigcup_{\ell \in \hat{L}(G)} \ell.$$

In order to prove Theorem 1, we compare two sets in $(\mathbf{P}^2_{\mathbb{C}})^*$: the effective lines $\mathcal{E}(G)$, (see Sect. 2.8), and the limit set $\hat{L}(G)$ in $(\mathbf{P}^2_{\mathbb{C}})^*$, (see Definition 5). We prove that this two sets are equal, and because under the hypothesis of Theorem 1, the identity $\bigcup_{\ell \in \mathcal{E}(G)} \ell = \Lambda_K(G)$ is satisfied. This equality implies the result.

It is a well known result, that given a closed nowhere dense subset of $\mathbb{S}^2$, there is a Kleinian group $G \subset \mathrm{PSL}(2, \mathbb{C})$ such that $C \subset \Lambda(G)$. Moreover, $\Lambda(G)$ is nowhere dense and C is close (in some way) to $\Lambda(G)$ (see Bernard 1988, VIII, A.7.).

A generalization of this result is obtained in the following way: If $F \subset (\mathbf{P}^2_{\mathbb{C}})^*$ is a closed set such that $\bigcup_{\ell \in F} \ell \neq \mathbf{P}^2_{\mathbb{C}}$ then there is a complex Kleinian group $G \subset \mathrm{PSL}(3, \mathbb{C})$ which is conjugate to a subgroup of $\mathrm{PU}(2, 1)$ such that $\Lambda_K(G) \supset \bigcup_{\ell \in F} \ell$. In fact, since $\bigcup_{\ell \in F} \ell$ is a proper closed subset of $\mathbf{P}^2_{\mathbb{C}}$, we can choose a ball B contained in $\mathbf{P}^2_{\mathbb{C}} \setminus \bigcup_{\ell \in F} \ell$ and a subgroup G preserving the ball B which is conjugate to a lattice of $\mathrm{PU}(2, 1)$, then

$$\Lambda_K(G) = \mathbf{P}^2_{\mathbb{C}} \setminus B \supset \bigcup_{\ell \in F} \ell,$$

where the equality above, is obtained as a consequence of the main Theorem in Navarrete (2006).

We remark that in this generalization, the limit set $\Lambda_K(G)$ has non-empty interior, so it is not necessarily close to $\bigcup_{\ell \in F} \ell$. The following theorem is a more faithful generalization of the classical result.

Theorem 2 *Given $\epsilon > 0$ and a closed subset $C \subset (\mathbf{P}^2_{\mathbb{R}})^*$ such that C has at least three points in general position and $\bigcup_{\ell \in C} \ell \neq \mathbf{P}^2_{\mathbb{C}}$, then there is a complex Kleinian group G_ϵ, such that the Hausdorff distance between $\hat{L}(G_\epsilon)$ and C is smaller than ϵ.*

The outline of the proof is: given a closed subset $C \in (\mathbf{P}^2_{\mathbb{R}})^*$, there is a finite set F with an even pair of elements such that the Hausdorff distance $d_H(F, C) < \epsilon/2$. Then we show that there is a family of Schottky type groups $G_\epsilon \subset \mathrm{PSL}(3, \mathbb{R})$ such that the Hausdorff distance $d_H(\hat{L}(G_\epsilon), F) < \epsilon/2$ and this implies that $d_H(\hat{L}(G_\epsilon), C) < \epsilon$. It is shown that $\bigcup_{\ell \in \hat{L}(G_\epsilon)} \ell \neq \mathbf{P}^2_{\mathbb{C}}$, and by Theorem 1, the Kulkarni limit set and $\bigcup_{\ell \in \hat{L}(G_\epsilon)} \ell$ coincide.

2 Preliminaries

2.1 The Complex Projective Plane

The *complex projective plane* $\mathbf{P}^2_{\mathbb{C}}$ is defined as the equivalence classes of the following equivalence relation in $\mathbb{C}^3 \setminus \{(0, 0, 0)\}$: for $(x, y, z), (u, v, w) \in \mathbb{C}^3 \setminus \{(0, 0, 0)\}$,

$$(x, y, z) \sim (u, v, w) \iff (u, v, w) = \alpha(x, y, z) \text{ for some } \alpha \in \mathbb{C} \setminus \{0\}.$$

$\mathbf{P}^2_{\mathbb{C}}$ is a 2-dimensional complex compact connected manifold. We denote by $[\ ]: \mathbb{C}^3 \setminus \{(0,0,0)\} \to \mathbf{P}^2_{\mathbb{C}}$ the canonical projection, so we write $[\mathbf{x}] = [x : y : z]$ for its corresponding projection to $\mathbf{P}^2_{\mathbb{C}}$, whenever $\mathbf{x} = (x, y, z)$. For the standard basis of $\mathbb{C}^3$: $\mathcal{B} = \{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ we write its projection $[\mathbf{e_j}]$ as $\mathbf{e_j}$ for $j = 1, 2, 3$.

We will say that $\ell \subset \mathbf{P}^2_{\mathbb{C}}$ is a complex projective line or line, for short, if the inverse image of the projection $[\ell]^{-1} \cup \{(0,0,0)\}$ is a 2-dimensional complex subspace of $\mathbb{C}^3$. Whenever $\mathbf{p}$ and $\mathbf{q}$ are points in $\mathbf{P}^2_{\mathbb{C}}$, we will denote the line through them as $\overleftrightarrow{\mathbf{p}, \mathbf{q}}$.

2.2 The Space of Lines of the Complex Projective Plane, $(\mathbf{P}^2_{\mathbb{C}})^*$

The space $(\mathbf{P}^2_{\mathbb{C}})^*$ is the space of complex projective lines $\ell \subset \mathbf{P}^2_{\mathbb{C}}$. This space can be identified with the complex projective plane $\mathbf{P}^2_{\mathbb{C}}$ as follows: the complex projective line ℓ with equation $Ax + By + Cz = 0$ is identified with the point $[A : B : C] \in \mathbf{P}^2_{\mathbb{C}}$, and sometimes we write $\ell = [A : B : C]$, by abuse of notation.

2.3 PSL(3, $\mathbb{C}$) and its Action on $\mathbf{P}^2_{\mathbb{C}}$ and $(\mathbf{P}^2_{\mathbb{C}})^*$

The transformations of $\mathbf{P}^2_{\mathbb{C}}$ are the elements in PSL(3, $\mathbb{C}$) where

$$\mathrm{PSL}(3, \mathbb{C}) = \mathrm{SL}(3, \mathbb{C}) \big/ \{\mathrm{Id},\ \omega\mathrm{Id},\ \omega^2\mathrm{Id}\},$$

being $\{1, \omega, \omega^2\}$ the cubic roots of unity and SL(3, $\mathbb{C}$) the group of 3×3-matrices with determinant equal one. PSL(3, $\mathbb{C}$) is a Lie group that acts transitively, faithfully and by biholomorphisms on $\mathbf{P}^2_{\mathbb{C}}$. The action is given as follows:

$$g \cdot [\mathbf{x}] = [\mathbf{g}(\mathbf{x})], \tag{1}$$

with $\mathbf{x} \in \mathbb{C}^3 \setminus \{(0,0,0)\}$ and $\mathbf{g} \in \mathrm{SL}(3, \mathbb{C})$.

An action of the elements in PSL(3, $\mathbb{C}$) can be also defined in the space of complex projective lines, $(\mathbf{P}^2_{\mathbb{C}})^*$: If $\ell = [A : B : C] \in (\mathbf{P}^2_{\mathbb{C}})^*$, and $g \in \mathrm{PSL}(3, \mathbb{C})$ then

$$g \cdot \ell = (\mathbf{g}^{-1})^T \begin{pmatrix} A \\ B \\ C \end{pmatrix}. \tag{2}$$

2.4 Pseudo-Projective Transformations

Throughout this paper $\mathbb{F}$ will denote either $\mathbb{R}$ or $\mathbb{C}$, unless otherwise stated. If $M_{3\times 3}(\mathbb{F})$ is the space of 3×3-matrices with entries in $\mathbb{F}$ equipped with the standard topology, the quotient space

$$\left(M_{3\times 3}(\mathbb{F}) \setminus \{\bar{\mathbf{0}}\}\right) \big/ \left(\mathbb{F} \setminus \{0\}\right), \tag{3}$$

is the *space of pseudo-projective maps of* $\mathbf{P}^2_{\mathbb{F}}$. It is naturally identified with the projective space $\mathbf{P}^8_{\mathbb{F}}$. $M_{3\times 3}(\mathbb{F}) \setminus \{\bar{\mathbf{0}}\}$ is a compactification of the open dense $\mathbb{F}^*$-invariant subset

GL(3, $\mathbb{F}$), and the space of pseudo-projective transformations is a compactification of PSL(3, $\mathbb{F}$).

If $\mathbf{s} \in M_{3\times 3}(\mathbb{F})$, then $[\mathbf{s}]$ denotes the equivalence class of the matrix $\mathbf{s}$ in the space of pseudo-projective transformations of $\mathbf{P}^2_{\mathbb{F}}$. And if S is a pseudo-projective transformation, a lift of S will be a matrix $\mathbf{s} \in M_{3\times 3}(\mathbb{F})\backslash\{\bar{\mathbf{0}}\}$ whenever $[\mathbf{s}] = S$.

The lift of a pseudo-projective transformation S induces a non-zero linear transformation $\mathbf{s} : \mathbb{F}^3 \to \mathbb{F}^3$ which is not necessarily invertible. Let $\mathrm{Ker}(\mathbf{s}) \subsetneq \mathbb{F}^3$ be its kernel and $\mathrm{Ker}(S) := [\mathrm{Ker}(\mathbf{s})]$ its projectivization in $\mathbf{P}^2_{\mathbb{F}}$. If $\mathrm{Ker}(\mathbf{s}) = \{(0, 0, 0)\}$, then $\mathrm{Ker}(S) := \varnothing$.

2.5 Complex Kleinian Groups

In Kulkarni (1978), the author defined a limit set for a group acting on very general topological spaces. His definition was used in this theory since its origin. We use the definition of limit set specifically for subgroups of PSL(3, $\mathbb{C}$) acting on $\mathbf{P}^2_{\mathbb{C}}$.

Let G be a subgroup of PSL(3, $\mathbb{C}$), we introduce three closed G-invariant subsets of $\mathbf{P}^2_{\mathbb{C}}$.

$L_0(G)$ the closure of points in $\mathbf{P}^2_{\mathbb{C}}$ with infinite isotropy group,
$L_1(G)$ the closure of accumulation points of the orbits of points in $\mathbf{P}^2_{\mathbb{C}}\backslash L_0(G)$,
$L_2(G)$ the closure of accumulation points of G-orbits of compact subsets contained in $\mathbf{P}^2_{\mathbb{C}} - (L_0(G) \cup L_1(G))$.

Definition 1 The union $L_0(G) \cup L_1(G) \cup L_2(G)$ will be called the *Kulkarni limit set* and it is denoted $\Lambda_K(G)$. The complement of this union $\Omega_K(G) := \mathbf{P}^2_{\mathbb{C}} - \Lambda_K(G)$ is the *discontinuity region* of G.

We say that G is a *complex Kleinian group* whenever $\Omega_K(G) \neq \varnothing$.

Proposition 1 *Let G be a subgroup of PSL(3, $\mathbb{C}$) acting in $\mathbf{P}^2_{\mathbb{C}}$. G equipped with the compact-open topology. Then $L_0(G)$, $L_1(G)$, $L_2(G)$, $\Lambda_K(G)$, $\Omega_K(G)$ are G-invariant and the action of G is properly discontinuous on $\Omega_K(G)$.*

It has also been proved in Barrera et al. (2011) that when G acts on $\mathbf{P}^2_{\mathbb{C}}$ without fixed points nor invariant lines, $\Omega_K(G)$ is the maximal open subset where the action is properly discontinuous.

Notation *When $G = \langle g\rangle$ is a cyclic group, we write $L_0(g)$ for $L_0(G)$, $\Lambda_K(g)$ for $\Lambda_K(G)$, etc.*

2.6 Kulkarni Limit Set of Elements in PSL(3, $\mathbb{C}$)

The transformations of PSL(3, $\mathbb{C}$) are classified as elliptic, parabolic or loxodromic elements. If an element of PSL(3, $\mathbb{C}$) is in any of these classes, there are specific Jordan canonical forms that the element might have. Navarrete (2008) studied this classification and the Kulkarni limit set for the canonical forms that different elements in PSL(3, $\mathbb{C}$) can have.

We can find in (Navarrete 2008, Proposition 4.3) that an element $g \in \mathrm{PSL}(3,\mathbb{C})$ is elliptic if and only if g has a lift $\mathbf{g} \in \mathrm{SL}(3,\mathbb{C})$ such that $\mathbf{g}$ is diagonalizable and every eigenvalue is an unitary complex number.

Therefore if an element of $\mathrm{PSL}(3,\mathbb{C})$ is elliptic, its lift $\mathbf{g}$ to $\mathrm{SL}(3,\mathbb{C})$ is conjugate to the next matrix:

$$\mathbf{h} = \begin{pmatrix} e^{2\pi i\alpha} & 0 & 0 \\ 0 & e^{2\pi i\beta} & 0 \\ 0 & 0 & e^{2\pi i\gamma} \end{pmatrix}, \tag{4}$$

where either $\mathbf{h}$ has finite or infinite order.

Remark 1 It is proved in (Navarrete 2008, Proposition 4.7) that when $\mathbf{h}$ has finite order the Kulkarni limit set is empty, and when it has infinite order, the Kulkarni limit set is $\mathbf{P}^2_{\mathbb{C}}$.

Analogously, an element in $\mathrm{PSL}(3,\mathbb{C})$ is parabolic if the Kulkarni limit set is equal to a single complex line. There are three different types of lift for a parabolic element:

$$\mathbf{f_1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{f_2} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{f_3} = \begin{pmatrix} e^{2\pi it} & 1 & 0 \\ 0 & e^{2\pi it} & 0 \\ 0 & 0 & e^{-4\pi it} \end{pmatrix}, \tag{5}$$

with $e^{2\pi it} \neq 1$.

Remark 2 In (Navarrete 2008, Proposition 5.4) it is shown that the Kulkarni limit set $\Lambda_K(\mathbf{f_1})$ is the line consisting of all the fixed points of $\mathbf{f_1}$, $\overleftrightarrow{\mathbf{e_1}, \mathbf{e_3}}$. The Kulkarni limit set of $\Lambda_K(\mathbf{f_2})$ is the unique $\mathbf{f_2}$-invariant complex line $\overleftrightarrow{\mathbf{e_1}, \mathbf{e_2}}$, and $\Lambda_K(\mathbf{f_3})$ is the line determined by the two fixed points, $\overleftrightarrow{\mathbf{e_1}, \mathbf{e_3}}$.

While for loxodromic elements, we can say that $g \in \mathrm{PSL}(3,\mathbb{C})$ is loxodromic if and only if g has a lift $\mathbf{g} \in \mathrm{SL}(3,\mathbb{C})$ with at least two eigenvalues of different module, (Navarrete 2008, Proposition 6.7). The lift that a loxodromic element can have is one of the four matrices below, when λ and μ are complex numbers different from zero.

$$\underset{|\lambda| \neq 1}{\mathbf{g_1} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}}, \quad \underset{\substack{\lambda \neq \mu \\ |\lambda| = |\mu| \neq 1}}{\mathbf{g_2} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & (\lambda\mu)^{-1} \end{pmatrix}}, \quad \underset{|\lambda| \neq 1}{\mathbf{g_3} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}}, \quad \underset{\substack{|\lambda_1| < |\lambda_2| < |\lambda_3| \\ \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}.}}{\mathbf{g_4} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}}. \tag{6}$$

Remark 3 In different propositions in (Navarrete 2008, Section 6) the author proves which is the Kulkarni limit set in each case. The transformation $\mathbf{g_1}$ is called complex homothety and $\Lambda_K(\mathbf{g_1})$ is $\{\mathbf{e_3}\} \cup \overleftrightarrow{\mathbf{e_1}, \mathbf{e_2}}$. The transformation $\mathbf{g_2}$ is called screw, its limit set is $\Lambda_K(\mathbf{g_2}) = \{\mathbf{e_3}\} \cup \overleftrightarrow{\mathbf{e_1}, \mathbf{e_2}}$. For $\mathbf{g_3}$, called loxoparabolic, the limit set is the line where the transformation acts as a parabolic element $\Lambda_K(\mathbf{g_3}) = \overleftrightarrow{\mathbf{e_1}, \mathbf{e_2}} \cup \overleftrightarrow{\mathbf{e_1}, \mathbf{e_3}}$. For the last transformation, a strongly loxodromic element, $\Lambda_K(\mathbf{g_4}) = \overleftrightarrow{\mathbf{e_1}, \mathbf{e_2}} \cup \overleftrightarrow{\mathbf{e_2}, \mathbf{e_3}}$, the complex lines determined by the repelling and saddle point and the attracting and saddle point.

2.7 Schottky Type Groups and the Conze–Guivarc'h Limit Set

Schottky groups are very useful groups for showing many different properties of the limit sets of groups acting on spaces.

We work with a type of groups introduced by Tits in (1972). Conze and Guivarc'h (2000) pick up the definition of Schottky type group and they study the ergodic properties of the limit set denoted by $L(G)$ that we now know as the Conze and Guivarc'h limit set.

The main difference with the classical Schottky groups is that in this case we do not ask the transformations to pair the circles in a way that the exterior of a compact set is sent exactly to the interior of another compact subset, it is enough for the image to be contained in the interior of the other compact subset.

Definition 2 Let (X, δ) be a complete metric space. A group Γ of homeomorphisms of X, generated by a finite symmetric set Σ (namely, $a^{-1} \in \Sigma$ for all $a \in \Sigma$) is called a group of Schottky type if there exists $\{C_a\}_{a\in\Sigma}$ a family of compact subsets of X, and a point $p \in X$ such that $p \notin \cup_{a\in\Sigma} C_a$ and $a(p) \in C_a$ for all $a \in \Sigma$, and the following conditions are satisfied:

(1) for $a, b \in \Sigma$, $C_a \cap C_b = \varnothing$ if $a \neq b$;
(2) for $a, b \in \Sigma$, $a(C_b) \subset Int(C_a)$, except when $ab = e$;
(3) for all sequences $\{a_n\}$ such that $a_n \neq a_{n+1}^{-1}$ for all $n > 1$, the diameter of $a_1 a_2 \cdots a_n(C_{a_{n+1}})$ tends to zero, when n tends to infinity.

The fact that there is a point p outside every compact set together with property (2) of the previous definition guarantee that the group generated by Σ is free and discrete (Conze and Guivarc'h 2000, Proposition 5.2).

In the context of Schottky type groups there is a definition of a convex set, used in Conze and Guivarc'h (2000):

Definition 3 A closed subset C of $\mathbf{P}^2_{\mathbb{F}}$ is said to be convex if it is contained in the complement of a projective hyperplane H and it is convex as a subset of the affine space $\mathbf{P}^2_{\mathbb{F}} - H$.

Following Conze and Guivarc'h (2000), and for making the notation easier, we introduce the next definition.

Notation *We will say that a set of homeomorphisms of a metric space X satisfies condition (S^+) if items (1) and (2) of Definition 2 are satisfied.*

Definition 4 A matrix $A \in \mathrm{GL}(3, \mathbb{F})$ is said to be proximal if it has one and only one eigenvalue with modulus larger than the modulus of all the other eigenvalues. We will call that eigenvalue as λ_A.

For a proximal matrix A, the vector $v_A \in \mathbb{F}^3$ will denote the corresponding eigenvector to the eigenvalue λ_A, and is called the dominant eigenvector of A.

Proposition 2 *Let A be a proximal transformation, being* λ_A *the eigenvalue of A with greater norm than the other eigenvalues. We define*

$$H_A^- = \{\omega \in \mathbb{C}^3 : \lambda_A^{-n} A^n \omega \to 0 \text{ as } n \to \infty\}.$$

Let S be the pseudo-projective limit of the positive powers of A. Then

$$Ker(S) = [H_A^-],$$

where $[H_A^-]$ denotes the projectivization of the vector subspace H_A^-.

Proof Let A be as $\mathbf{g_4}$ in Eq. (6). Then

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The kernel of S, is $\text{Ker}(S) = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x, y \in \mathbb{C} \right\}$. Now,

$$H_A^- = \left\{ \omega = \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \lambda_3^{-n} A^n \omega \to 0 \right\}.$$

$$\lambda_3^{-n} A^n \omega = \lambda_3^{-n} \begin{pmatrix} \lambda_1^n x \\ \lambda_2^n y \\ \lambda_3^n z \end{pmatrix} = \begin{pmatrix} \left(\frac{\lambda_1}{\lambda_3}\right)^n x \\ \left(\frac{\lambda_2}{\lambda_3}\right)^n y \\ z \end{pmatrix}.$$

Then it is clear that

$$\text{Ker}(S) \subset H_A^-.$$

And because $\dim(\text{Ker}(S)) = \dim(H_A^-) = 2$, we conclude that $\text{Ker}(S) = H_A^-$.

There are other possibilities for A to be a proximal transformation, for example, if A is either as $\mathbf{g_1}$ or $\mathbf{g_3}$ in Eq. (6) with $|\lambda| < 1$. It is not hard to check the Proposition 2 is still true for different types of proximal elements. □

For the reader's convenience we give a brief introduction to the limit set presented in Conze and Guivarc'h (2000). Also, we provide useful examples in order to compare the limit set of Conze and Guivarc'h and the limit set in the sense of Kulkarni.

We restate the Propositions 5.9 and 5.10 of Conze and Guivarc'h (2000).

Proposition 3 *Let* $\hat{\Sigma}$ *be a family of pairs* $\{(a, C_a) | a \in \Sigma\}$ *where* Σ *is a set of projective transformations and* $a \in \Sigma$ *with eigenvector* $a^+ \in C_a$ *associated to the greatest eigenvalue and where* C_a *are disjoint compact convex sets such that* $[H_b^-] \cap C_a = \varnothing$

if $b \neq a^{-1}$. *Then for all sufficiently large n the family* $\widehat{\Sigma}_n = \{(a^n, C_a) | a \in \Sigma\}$ *satisfies condition* (S^+).

Moreover, under the same hypothesis if the family $\hat{\Sigma}$ *satisfies condition* (S^+)*, then condition (3) as in Definition 2 also holds.*

Remark 4 If b is a proximal element and S is a pseudo-projective limit of the positive powers of b, then $[H_b^-] = [\text{Ker}(S)]$. And the condition $[H_b^-] \cap C_a \neq \varnothing$ in the Proposition 3 can be restated as $[\text{Ker}(S)] \cap C_a \neq \varnothing$.

According to the classification of transformations of $\text{PSL}(3, \mathbb{C})$ given in Navarrete (2008), can be deduced that a proximal transformation is loxodromic, but the converse is not true. Moreover, every proximal element has an attracting fixed point in $\mathbf{P}^2_{\mathbb{F}}$, then we have the following definition.

Definition 5 (*Conze and Guivarc'h limit set*) Let G be a subgroup of $\text{GL}(3, \mathbb{F})$ and consider its action on $\mathbf{P}^2_{\mathbb{F}}$. We denote by $L(G)$ the closure of the subset of $\mathbf{P}^2_{\mathbb{F}}$ consisting of all the attracting fixed points of proximal elements of G.

We emphasize that $L(G)$ is always a G-invariant subset of $\mathbf{P}^2_{\mathbb{F}}$ and when the G-action is irreducible (i.e it does not exist any proper subspace of $\mathbf{P}^2_{\mathbb{F}}$ invariant under the action of a subgroup of finite index in G) and when G has a proximal element, then $L(G)$ is a minimal subset for this G-action.

Example 1 Consider the strongly loxodromic transformation $\mathbf{g_4}$ in Eq. (6) acting on $\mathbf{P}^2_{\mathbb{C}}$. The Kulkarni limit set is $\Lambda_K(\mathbf{g_4}) = \overleftrightarrow{\mathbf{e_1}, \mathbf{e_2}} \cup \overleftrightarrow{\mathbf{e_2}, \mathbf{e_3}}$, (Remark 3). It is not hard to check that $L(\mathbf{g_4})$ is equal to $\{\mathbf{e_1}, \mathbf{e_3}\}$.

Observe that $L(\mathbf{g_4}) \subset L_0(\mathbf{g_4}) \subset \Lambda_K(\mathbf{g_4})$. The action of G on $\mathbf{P}^2_{\mathbb{C}} - L(\mathbf{g_4})$ is not properly discontinuous, while the action of G on $\mathbf{P}^2_{\mathbb{C}} - \Lambda_K(\mathbf{g_4})$ is.

2.8 Two Sets of Lines

In the article Barrera et al. (2016), the authors introduced the concept of *effective lines* of a discrete group G. First, if G' is the set $\{S$ pseudo-projective map of $\mathbf{P}^2_{\mathbb{C}}$: S is a cluster point of G$\}$, then $\mathcal{E}(G) \subset (\mathbf{P}^2_{\mathbb{C}})^*$ is the set $\{\ell \subset \mathbf{P}^2_{\mathbb{C}} : \ell = \text{Ker}(S),$ for some $S \in G'\}$ where ℓ is a complex line in $\mathbf{P}^2_{\mathbb{C}}$.

In Barrera et al. (2011), the authors introduce the set $E(G)$ as the subset of $(\mathbf{P}^2_{\mathbb{C}})^*$ consisting of all the complex lines ℓ for which there exists an element $g \in G$ such that $\ell \subset \Lambda_K(g)$.

We prove the following proposition:

Proposition 4 *Let G be a discrete subgroup of PSL*$(3, \mathbb{C})$*, with at least three lines in general position in* $E(G)$. *Then*

$$\overline{E(G)} = \mathcal{E}(G). \tag{7}$$

Proof First we prove $E(G) \subset \mathcal{E}(G)$. Let ℓ be a line in $E(G)$, then there is an element $g \in G$ such that $\ell \subset \Lambda_K(g)$. Each line in the Kulkarni limit set is the kernel of the pseudo-projective transformation obtained as the limit of g^n or g^{-n}, with $n \in \mathbb{N}$. It follows that $\overline{E(G)} \subset \mathcal{E}(G)$ because $\mathcal{E}(G)$ is closed, (Barrera et al. 2016, Proposition 4.2).

Conversely, let $\ell \in \mathcal{E}(G)$, then $\ell = \text{Ker}(S)$ where $S = \lim_{n\to\infty} g_n$, for some sequence $(g_n) \subset G$. Take $\ell_0 \subset \Lambda_K(g_0)$ a line in $E(G)$ not passing through the point $\text{Im}(S)$. By (Barrera et al. 2011, Lemma 3.2(3)) the sequence $g_n^{-1} \cdot \ell_0$ converges to $\text{Ker}(S) = \ell$, where for each $n \in \mathbb{N}$, $g_n^{-1} \cdot \ell_0 \subset \Lambda_K(g_n^{-1} g_0 g_n)$ is in $E(G)$. □

3 The Limit Set in $(\mathbf{P}^2_{\mathbb{C}})^*$

In this section we extend the Definition 4 to work with every type of elements in $\text{PSL}(3, \mathbb{C})$. We propose the following definition.

Definition 6 Let us consider $G \subset \text{PSL}(3, \mathbb{C})$ acting on $(\mathbf{P}^2_{\mathbb{C}})^*$. We say that $\mathbf{q} \in (\mathbf{P}^2_{\mathbb{C}})^*$ is a limit point of G if there exists an open subset $U \subset (\mathbf{P}^2_{\mathbb{C}})^*$ and there exists a sequence $\{g_n\} \subset G$, $g_n \neq g_m$ if $n \neq m$, such that for every $\mathbf{p} \in U$

$$\lim_{n\to\infty} g_n \cdot \mathbf{p} = \mathbf{q} \tag{8}$$

The set of limit points will be called the limit set, denoted by $\hat{L}(G)$.

Example 2 If $g \in \text{PSL}(3, \mathbb{C})$ is a strongly loxodromic element, then without loss of generality we can assume that g is induced by the matrix $\mathbf{g_4}$ in Eq. (6).

When we consider the element g acting in $(\mathbf{P}^2_{\mathbb{C}})^*$ we notice that the complex lines

$$\begin{aligned}
\ell_1 &= \{[x : y : z] \in \mathbf{P}^2_{\mathbb{C}} : x = 0\},\\
\ell_2 &= \{[x : y : z] \in \mathbf{P}^2_{\mathbb{C}} : y = 0\},\\
\ell_3 &= \{[x : y : z] \in \mathbf{P}^2_{\mathbb{C}} : z = 0\},
\end{aligned}$$

correspond to the eigenvectors of

$$(\mathbf{g}^{-1})^T = \begin{pmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & \frac{1}{\lambda_2} & 0 \\ 0 & 0 & \frac{1}{\lambda_3} \end{pmatrix}.$$

Hence, ℓ_1, ℓ_2, ℓ_3 are the fixed points for the action of g on $(\mathbf{P}^2_{\mathbb{C}})^*$. In fact, ℓ_1 is an attracting fixed point, because for every $\eta \in U_1 = (\mathbf{P}^2_{\mathbb{C}})^* \setminus \overleftrightarrow{\ell_2, \ell_3}$, $g^n \cdot \eta \to \ell_1$ as $n \to \infty$; and ℓ_3 is a repelling fixed point because for every $\eta \in U_3 = (\mathbf{P}^2_{\mathbb{C}})^* \setminus \overleftrightarrow{\ell_1, \ell_2}$, $g^{-n} \cdot \eta \to \ell_1$ as $n \to \infty$. Where $\overleftrightarrow{\ell_\mathbf{j}, \ell_\mathbf{k}}$ denotes the projective line passing through the points $\ell_\mathbf{j}, \ell_\mathbf{k} \in (\mathbf{P}^2_{\mathbb{C}})^*$.

Therefore, ℓ_1 and ℓ_3 are the only limit points, according to Definition 6, for the cyclic group generated by g, and it is not hard to check that

$$\Lambda_K(g) = \ell_1 \cup \ell_3 = \bigcup_{\ell \in \hat{L}(g)} \ell.$$

Lemma 1 *If $G = \langle g \rangle \subset PSL(3, \mathbb{C})$ is a cyclic subgroup then:*

(i) $\Lambda_K(G) = \bigcup_{\ell \in \hat{L}(G)} \ell$ *whenever g is neither a complex homothety nor a screw.*
(ii) If g is either a complex homothety or a screw then $\Lambda_K(G) \supsetneq \bigcup_{\ell \in \hat{L}(G)} \ell$.

Proof It is enough to verify the Lemma for the elements of different type, see Sect. 2.6.

(i) If $g \in \mathrm{PSL}(3, \mathbb{C})$ is a loxoparabolic transformation, g has a lift in $\mathrm{SL}(3, \mathbb{C})$ whose Jordan canonical form is given by the matrix $\mathbf{g_3}$ in Eq. (6). g acts in $(\mathbf{P}^2_{\mathbb{C}})^*$ as we said in Eq. (2).
For any $[A : B : C]$ in the open subset U_1 of $(\mathbf{P}^2_{\mathbb{C}})^*$, where

$$U_1 = \{[A : B : C] \in (\mathbf{P}^2_{\mathbb{C}})^* : A \neq 0\}, \tag{9}$$

and for the sequence $\{g^n\}_{n \in \mathbb{N}}$, the sequence of lines in $(\mathbf{P}^2_{\mathbb{C}})^*$ given by

$$\left(\mathbf{g}^{-n}\right)^T \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} \lambda^{-n} & 0 & 0 \\ -n\lambda^{-(n+1)} & \lambda^{-n} & 0 \\ 0 & 0 & \lambda^{2n} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} \tag{10}$$

is projectively the same as the sequence

$$\frac{\lambda^{n+1}}{n} \left(\mathbf{g}^{-n}\right)^T \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} \frac{\lambda}{n} A \\ -A + \frac{\lambda}{n} B \\ \frac{\lambda^{3n+1}}{n} C \end{pmatrix}, \tag{11}$$

and this last sequence converges to the line given by $[0 : 1 : 0]$.
Now, take the action of g^{-1} in $(\mathbf{P}^2_{\mathbb{C}})^*$ and let U_3 be the open subset of $(\mathbf{P}^2_{\mathbb{C}})^*$ defined by $\{[A : B : C] \in (\mathbf{P}^2_{\mathbb{C}})^* : C \neq 0\}$. The sequence of lines in $(\mathbf{P}^2_{\mathbb{C}})^*$ given by:

$$\left(\mathbf{g}^{n}\right)^T \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} \lambda^{n} & 0 & 0 \\ n\lambda^{n-1} & \lambda^{n} & 0 \\ 0 & 0 & \lambda^{-2n} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} \tag{12}$$

is projectively equivalent to the sequence:

$$\lambda^{2n} \left(\mathbf{g}^{n}\right)^T \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} \lambda^{3n} A \\ n\lambda^{3n-1} A + \lambda^{3n} B \\ C \end{pmatrix}, \tag{13}$$

converges to the line $[0 : 0 : 1]$, whenever $[A : B : C]$ is in U_3.
So $\hat{L}(G) = \{\ell_2, \ell_3\}$, therefore, the lemma is true for loxoparabolic elements.
For the other elements whose canonical Jordan forms are defined in Sect. 2.6, the analysis is very similar to the previous and we summarize it in the Table:

g acting in $\mathbf{P}^2_{\mathbb{C}}$	$\Lambda_K(g)$	g acting in $(\mathbf{P}^2_{\mathbb{C}})^*$	$\hat{L}(g)$	Open subset
Loxodromic elements				
Loxoparabolic $\mathbf{g} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}$ $\lvert\lambda\rvert > 1$	$\overleftrightarrow{\mathbf{e_1e_2}} \cup \overleftrightarrow{\mathbf{e_1e_3}}$	$(\mathbf{g}^{-1})^T = \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ -\lambda^{-2} & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^2 \end{pmatrix}$	$\{\ell_2, \ell_3\}$	U_1 and U_3 resp.
Stronglyloxodromic $\mathbf{g} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ $\lvert\lambda_1\rvert < \lvert\lambda_2\rvert < \lvert\lambda_3\rvert$	$\overleftrightarrow{\mathbf{e_1e_2}} \cup \overleftrightarrow{\mathbf{e_2e_3}}$	$(\mathbf{g}^{-1})^T = \begin{pmatrix} \lambda_1^{-1} & 0 & 0 \\ 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & \lambda_3^{-1} \end{pmatrix}$	$\{\ell_1, \ell_3\}$	U_1 and U_3 resp.
Parabolic elements				
$\mathbf{g} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\overleftrightarrow{\mathbf{e_1e_3}}$	$(\mathbf{g}^{-1})^T = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\{\ell_2\}$	U_1
$\mathbf{g} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\overleftrightarrow{\mathbf{e_1e_2}}$	$(\mathbf{g}^{-1})^T = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$	$\{\ell_3\}$	U_1
$\mathbf{g} = \begin{pmatrix} e^{2\pi it} & 1 & 0 \\ 0 & e^{2\pi it} & 0 \\ 0 & 0 & e^{-4\pi it} \end{pmatrix}$ $e^{-2\pi it} \neq 1$	$\overleftrightarrow{\mathbf{e_1e_3}}$	$(\mathbf{g}^{-1})^T = \begin{pmatrix} e^{-2\pi it} & 0 & 0 \\ -e^{-4\pi it} & e^{-2\pi it} & 0 \\ 0 & 0 & e^{4\pi it} \end{pmatrix}$	$\{\ell_2\}$	U_1
Elliptic elements				
$\mathbf{g} = \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix}$ g has infinite order	$\mathbf{P}^2_{\mathbb{C}}$	$(\mathbf{g}^{-1})^T = \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix}$	$(\mathbf{P}^2_{\mathbb{C}})^*$	$(\mathbf{P}^2_{\mathbb{C}})^*$
$\mathbf{g} = \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix}$ g has finite order	$\varnothing$	$(\mathbf{g}^{-1})^T = \begin{pmatrix} e^{-i\theta_1} & 0 & 0 \\ 0 & e^{-i\theta_2} & 0 \\ 0 & 0 & e^{-i\theta_3} \end{pmatrix}$	$\varnothing$	

(ii) If $g \in \mathrm{PSL}(3,\mathbb{C})$ is a screw, then we can assume that g is induced by a matrix of the form $\mathbf{g_2}$ in Eq. (6).
It is not hard to check that $g^n \cdot \ell \to \ell_3$ as $n \to \infty$ for every $\ell \in U_3$, so $\hat{L}(g) = \{\ell_3\}$. It follows, from Sect. 2.6, that $\Lambda_K(g) = \overleftrightarrow{\mathbf{e_1}, \mathbf{e_2}} \cup \{\mathbf{e_3}\}$. Hence

$$\bigcup_{\ell \in \hat{L}(g)} \ell \subsetneq \Lambda_K(G).$$

The case when g is a complex homothety is analogous to the previous one and we omit it. □

If G is a discrete subgroup of $\mathrm{PSL}(3,\mathbb{C})$, we recall that $\mathcal{E}(G)$ denotes the set of complex lines, ℓ, for which there exists a sequence $(g_n) \subset G$ of distinct elements such that g_n converges to the pseudo-projective transformation S as $n \to \infty$, and $\ell = \mathrm{Ker}(S)$.

Proposition 5 *If $G \subset PSL(3,\mathbb{C})$ is a discrete subgroup then*

$$\mathcal{E}(G) = \hat{L}(G).$$

Proof Let ℓ be in $\mathcal{E}(G)$, thus there exists a sequence of distinct elements $(g_n) \subset G$ and a pseudo-projective transformation S, such that $g_n \to S$ as $n \to \infty$ uniformly on compact subsets of $\mathbf{P}^2_{\mathbb{C}} \setminus \mathrm{Ker}(S) = \mathbf{P}^2_{\mathbb{C}} \setminus \ell$. By (Barrera et al. 2011, Lemma 3.2), we can assume that there exists R pseudo-projective transformation, such that $g_n^{-1} \to R$ as $n \to \infty$ uniformly on compact subsets of $\mathbf{P}^2_{\mathbb{C}} \setminus \mathrm{Ker}(R)$. Moreover, if η is a complex line in the open set $U = \{\eta \in (\mathbf{P}^2_{\mathbb{C}})^* : \mathrm{Im}(S) \text{ does not lie on } \eta\}$ then $g_n^{-1} \cdot \eta \to \mathrm{Ker}(S)$ as $n \to \infty$.

Conversely, let $\ell = [A : B : C] \in \hat{L}(G)$, so there is a non-empty open set $U \subset (\mathbf{P}^2_{\mathbb{C}})^*$ such that $g_n \cdot \eta \to \ell$ as $n \to \infty$ for every $\eta \in U$. If we use (Barrera et al. 2011, Lemma 3.2) for the sequence of projective transformations $[(\mathbf{g}_n^{-1})^T]$, we obtain a pseudo-projective transformation S such that

$$[(\mathbf{g}_n^{-1})^T] \to S \text{ as } n \to \infty \text{ uniformly on compact subsets of } \mathbf{P}^2_{\mathbb{C}} \setminus \mathrm{Ker}(S). \tag{14}$$

Moreover, the hypothesis that all lines η in the non-empty open set U satisfy that $g_n \cdot \eta \to \ell$ as $n \to \infty$ imply that $\mathrm{Im}(S)$ consists of one point. In fact, $\mathrm{Im}(S) = \{[A : B : C]\}$, so we can write $S = [\mathbf{s}]$, where

$$\mathbf{s} = \begin{pmatrix} \lambda A & \mu A & \nu A \\ \lambda B & \mu B & \nu B \\ \lambda C & \mu C & \nu C \end{pmatrix}, \quad \text{where } |\lambda| + |\mu| + |\nu| \neq 0.$$

It follows from (14) that

$$g_n^{-1} = [\mathbf{g}_n^{-1}] \to S' = [\mathbf{s}^T] \text{ as } n \to \infty \text{ uniformly on compact subsets of } \mathbf{P}^2_{\mathbb{C}} \setminus \mathrm{Ker}(S').$$

Moreover, $\mathrm{Ker}(S') = [\mathrm{Ker}(\mathbf{s}^T)] = \{[x : y : z] | Ax + By + Cz = 0\} = \ell$. □

Corollary 1 *If $G \subset PSL(3, \mathbb{C})$ is a discrete subgroup then $\hat{L}(G) \subset (\mathbf{P}^2_{\mathbb{C}})^*$ is a closed set.*

Proof The Proposition 5 implies that $\hat{L}(G) = \mathcal{E}(G)$, and (Barrera et al. 2016, Proposition 4.2) states that $\mathcal{E}(G) \subset (\mathbf{P}^2_{\mathbb{C}})^*$ is closed. □

Corollary 2 *Let G be a discrete subgroup of $PSL(3, \mathbb{C})$, and H subgroup of G, with $[G : H] < \infty$. Then $\hat{L}(H) = \hat{L}(G)$.*

Proof It is not hard to check that $\hat{L}(H) \subset \hat{L}(G)$. Let $\ell \in \hat{L}(G)$. Since $\hat{L}(G) = \mathcal{E}(G)$, there exists a sequence $(g_n) \subset G$ such that $g_n \to S$, S a pseudo-projective transformation, and $\mathrm{Ker}(S) = \ell$. As $[G : H] < \infty$, there exists $a \in G$ and $(h_n) \subset H$, with $h_n \neq h_m$ whenever $n \neq m$. With out loss of generality $g_n = ah_n$. If $R = \lim_{n\to\infty} h_n$, then $\mathrm{Ker}(R) = \mathrm{Ker}(S)$, this implies that $\mathcal{E}(H) = \hat{L}(H)$. □

Corollary 3 (Properties of limit set $\hat{L}(G)$) *Let G be a discrete subgroup of $PSL(3, \mathbb{C})$. Assume that G acts in $\mathbf{P}^2_{\mathbb{C}}$ without global fixed points nor invariant lines, and $\hat{L}(G)$ contains at least four elements, then:*

(i) *$\hat{L}(G)$ is a perfect set and it is the minimal closed set for the action of G on $(\mathbf{P}^2_{\mathbb{C}})^*$.*
(ii) *The G-orbit of any $\eta \in \hat{L}(G)$ is dense in $\hat{L}(G)$.*
(iii) *$\hat{L}(G)$ is the closure of the set of loxodromic fixed points, and if there are parabolic elements in G, then $\hat{L}(G)$ is the closure of the set of parabolic fixed points as well.*
(iv) *$\hat{L}(G) = (\mathbf{P}^2_{\mathbb{C}})^*$ or it has empty interior.*

Proof First, we prove (i). The Proposition 5 implies $\hat{L}(G) = \mathcal{E}(G)$, and Proposition 4 together with (Barrera et al. 2011, Theorem 1.3 (c)) implies the result.

The proof of (ii) and (iii) is a consequence of the minimality of $\hat{L}(G)$.

Now, we prove (iv). We notice that

$$\mathrm{Eq}(G) = \mathbf{P}^2_{\mathbb{C}} \setminus \bigcup_{\ell \in \mathcal{E}(G)} \ell = \mathbf{P}^2_{\mathbb{C}} \setminus \bigcup_{\ell \in \hat{L}(G)} \ell,$$

where the first equality is obtained by (Barrera et al. 2016, Corollary 4.5), and the second is obtained by Proposition 5 above. By applying (Barrera et al. 2016, Proposition 4.10) we see that there exists a loxodromic element in G.

Let us assume that U is a non-empty open subset of $\hat{L}(G)$. By (iii), there exists $\ell \in U$ where ℓ is an attracting fixed line for a loxodromic $g_0 \in G$. If $\varnothing \neq W$ is an open set contained in $(\mathbf{P}^2_{\mathbb{C}})^* \setminus \hat{L}(G)$ then there is $\eta \in W$ such that $g_0^n \cdot \eta \in U$ for all n large enough. This is a contradiction to the fact that $\hat{L}(G)$ is G-invariant. □

Having proved the properties of $\hat{L}(G)$ we can state the next theorem.

Theorem 1 *Let $G \leq PSL(3, \mathbb{C})$ be an infinite discrete subgroup acting in $\mathbf{P}^2_{\mathbb{C}}$ without fixed points nor invariant lines. Let $\hat{L}(G)$ be the limit set of G acting on $(\mathbf{P}^2_{\mathbb{C}})^*$, then*

$$\Lambda_K(G) = \bigcup_{\ell \in \hat{L}(G)} \ell.$$

Proof First we observe that $\Lambda_K(G) = \mathbf{P}^2_{\mathbb{C}} \setminus \mathrm{Eq}(G)$ by (Barrera et al. 2011, Theorem 1.3 (a)). Then $\mathbf{P}^2_{\mathbb{C}} \setminus \mathrm{Eq}(G) = \bigcup_{\ell \in \mathcal{E}(G)} \ell$, by (Barrera et al. 2016, Corollary 4.5). Finally, by Proposition 5, $\bigcup_{\ell \in \mathcal{E}(G)} \ell = \bigcup_{\ell \in \hat{L}(G)} \ell$. □

In the following example we present a group Γ with parabolic elements in which $\hat{L}(\Gamma)$ is identified with the classical limit set Λ for discrete subgroups of $\mathrm{PU}(2,1)$ acting in $\mathbf{H}^2_{\mathbb{C}}$.

Example 3 In Gusevskii and Parker (2003) the authors give a type-preserving representation ρ of the group $\mathrm{PSL}(2,\mathbb{Z})$ in $\mathrm{PU}(2,1)$. The image under ρ of the two generators of $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ generate a discrete subgroup in $\mathrm{PU}(2,1)$, $\rho(\Gamma)$. In Navarrete (2006), the author shows that the Kulkarni limit set of $\rho(\Gamma)$ is the set:

$$\Lambda_K(\Gamma) = \bigcup_{x \in \Lambda} \ell_x,$$

where ℓ_x is a tangent line to $\partial \mathbf{H}^2_{\mathbb{C}}$ in x.

Now, by Theorem 1, we show that $\hat{L}(\Gamma) = \{\ell_x \in (\mathbf{P}^2_{\mathbb{C}})^* : x \in \Lambda\}$.

4 Building the Subgroup Acting on $(\mathbf{P}^2_{\mathbb{C}})^*$

Given (X, d) a metric space, it is well known that the collection of compact subsets of X has a distance called the *Hausdorff distance*. For the convenience of the reader we recall the definition of this distance.

$$d_H(A, B) = \inf\{r > 0 | A \subset B_r \text{ and } B \subset A_r\}, \tag{15}$$

where A and B are compact subsets of X and $A_r = \{x \in X : d(x, A) < r\}$ is the r-neighborhood of A.

Lemma 2 *Given $\eta, \mu, \nu \in \mathbf{P}^2_{\mathbb{F}}$, there exists $g \in PSL(3, \mathbb{F})$ strongly loxodromic transformation satisfying the following:*

- (i) *η, μ and ν are fixed lines for g and $\hat{L}(g) = \{\eta, \mu\}$.*
- (ii) *For all neighborhood W such that $\overline{W} \subset \mathbf{P}^2_{\mathbb{F}} \setminus \overleftrightarrow{\mu, \nu}$, and any neighborhood U of η, exists $N \in \mathbb{N}$ such that $g^n \cdot W \subset U$ for $n > N$.*
- (iii) *For all neighborhood W such that $\overline{W} \subset \mathbf{P}^2_{\mathbb{F}} \setminus \overleftrightarrow{\eta, \nu}$, and any neighborhood V of μ, exists $N \in \mathbb{N}$ such that $g^{-n} \cdot W \subset V$ for $n > N$.*

The proof of this Lemma follows from (Barrera et al. 2011, Lemma 3.2).

Remark 5 Let η and μ be elements in $(\mathbf{P}^2_{\mathbb{F}})^*$, and let F be a finite subset of $(\mathbf{P}^2_{\mathbb{F}})^*$. Then, there exists $\epsilon > 0$ and a $\nu \in (\mathbf{P}^2_{\mathbb{F}})^*$ such that $\overleftrightarrow{\eta, \nu}$ does not intersect the balls with radio ϵ and center in $F \cup \{\mu\}$. And $\overleftrightarrow{\nu, \mu}$ does not intersect the closure of the balls with radio ϵ and center in $F \cup \{\eta\}$.

The next lemma illustrates the construction of the group of G_ϵ of Theorem 2, for the particular case when the closed subset $C \subset (\mathbf{P}^2_{\mathbb{R}})^*$ consists of four points.

Lemma 3 *Given* $F = \{\eta_1, \mu_1, \eta_2, \mu_2\} \subset (\mathbf{P}^2_{\mathbb{R}})^*$, *and* $\epsilon > 0$, *there exists a Schottky type group* G_ϵ *such that*

$$d_H(\hat{L}(G_\epsilon), F) < \epsilon \tag{16}$$

Proof Consider $\epsilon > 0$. Let U_i and V_i balls with center η_i and μ_i, respectively, and radio $0 < \epsilon' \leq \epsilon$ such that the U_1, U_2, V_1, V_2 are pairwise disjoint. Using Remark 5 there exists $\epsilon_1 > 0$ and a $\nu_1 \in (\mathbf{P}^2_{\mathbb{R}})^*$ such that $\overleftrightarrow{\eta_1, \nu_1}$ does not intersect the closure of the balls with radio ϵ_1 and center in $\{\eta_2, \mu_2\} \cup \{\mu_1\}$. And $\overleftrightarrow{\nu_1, \mu_1}$ does not intersect the closure of the balls with radio ϵ_1 and center in $\{\eta_2, \mu_2\} \cup \{\eta_1\}$.

Analogously, there exists $\epsilon_2 > 0$ and a $\nu_2 \in (\mathbf{P}^2_{\mathbb{R}})^*$ such that $\overleftrightarrow{\eta_2, \nu_2}$ does not intersect the balls with radio ϵ_2 and center in $\{\eta_1, \mu_1\} \cup \mu_2$. And $\overleftrightarrow{\nu_2, \mu_2}$ does not intersect the closure of the balls with radio ϵ_2 and center in $\{\eta_1, \mu_1\} \cup \{\eta_2\}$. We take $\epsilon_3 = \min\{\epsilon', \epsilon_1, \epsilon_2\}$.

Applying Lemma 2, there are strongly loxodromic transformations $g_1, g_2 \in PSL(3, \mathbb{R})$ such that η_i, μ_i, ν_i are fixed points for the transformation g_i for $i = 1, 2$ and $\hat{L}(g_i) = \{\eta_i, \mu_i\}$. Also, for every open subset W such that $\overline{W} \subset (\mathbf{P}^2_{\mathbb{R}})^* \setminus \overleftrightarrow{\mu_i, \nu_i}$ and any neighborhood U_i of η_i there exists $N_i \in \mathbb{N}$ such that for $n > N_i$, $g_i^n \cdot W \subset U_i$. Also for every open subset W such that $\overline{W} \subset (\mathbf{P}^2_{\mathbb{R}})^* \setminus \overleftrightarrow{\eta_i, \nu_i}$ and any neighborhood V_i if μ_i there exists $M_i \in \mathbb{N}$ such that for $n > M_i$, $g_i^{-n} \cdot W \subset V_i$, $i = 1, 2$. In particular, if we take U_i and V_i as balls with radio ϵ_3, we have the hypothesis of Proposition 3. So, for $N = \max\{N_1, N_2, M_1, M_2\}$, the group $G_\epsilon = \langle g_1^N, g_2^N \rangle$ is a Schottky type group. As $F \subset \hat{L}(G_\epsilon)$ and $\hat{L}(G_\epsilon) \subset \bigcup_{f \in F} B(f, \epsilon_3)$, it is not hard to check that $d_H(\hat{L}(G_\epsilon), F) < \epsilon_3 < \epsilon$. □

Lemma 4 *Given* F *a finite subset of points in* $(\mathbf{P}^2_{\mathbb{F}})^*$ *and* $\epsilon > 0$, *there exists a Schottky type group* G_ϵ *such that*

$$d_H(\hat{L}(G_\epsilon), F) < \epsilon \tag{17}$$

The proof of this lemma is analogous to the proof of Lemma 3.

Lemma 5 *Let* C *be a closed subset of* $(\mathbf{P}^2_{\mathbb{F}})^*$ *such that* $\bigcup_{\ell \in C} \ell \neq \mathbf{P}^2_{\mathbb{F}}$. *Then there exists* $\epsilon > 0$ *such that* $\overline{C}_\epsilon = \{\ell \in (\mathbf{P}^2_{\mathbb{F}})^* : d(\ell, C) \leq \epsilon\}$, *satisfies* $\bigcup_{\ell \in \overline{C}_\epsilon} \ell \neq \mathbf{P}^2_{\mathbb{F}}$.

Proof Let p be a point in $\mathbf{P}^2_{\mathbb{F}} \setminus \bigcup_{\ell \in C} \ell$, then there is a line $\mathcal{L}$ in $(\mathbf{P}^2_{\mathbb{F}})^*$ such that $\mathcal{L} \cap C = \varnothing$ and $p \in \bigcup_{\ell \in \mathcal{L}} \ell$. Then there exists $\epsilon > 0$ satisfying $\overline{N_\epsilon(C)} \cap \mathcal{L} = \varnothing$. Therefore, p is in $\mathbf{P}^2_{\mathbb{F}} \setminus \bigcup_{\ell \in \overline{N_\epsilon(C)}} \ell$. □

Now, we begin with the proof of Theorem 2.

Theorem 2 *Given* $\epsilon > 0$ *and a closed subset* $C \subset (\mathbf{P}^2_{\mathbb{R}})^*$ *such that* C *has at least three points in general position and* $\bigcup_{\ell \in C} \ell \neq \mathbf{P}^2_{\mathbb{C}}$, *there is a complex Kleinian group* G_ϵ, *such that the Hausdorff distance between* $\hat{L}(G_\epsilon)$ *and* C *is smaller than* ϵ.

Proof Consider $\epsilon > 0$. With out loss of generality, we choose F finite subset of C such that F has three points in general position and

$$d_H(F, C) < \epsilon/2. \tag{18}$$

By Lemma 4 there exists a Schottky type group G_ϵ such that

$$d_H(\hat{L}(G_\epsilon), F) < \epsilon/2. \tag{19}$$

From Eqs. (18) and (19) we have

$$d_H(\hat{L}(G_\epsilon), C) < \epsilon.$$

By Lemma 5, for any small enough ϵ we have the equality

$$\mathbf{P}^2_{\mathbb{C}} \neq \bigcup_{\ell \in \hat{L}(G_\epsilon)} \ell.$$

And by Theorem 1, $\Lambda_K(G_\epsilon) = \bigcup_{\ell \in \hat{L}(G_\epsilon)} \ell$. □

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